神戸学院法学第42巻第２号（2012年12月）

Classification of complex Hadamard matrices on pseudocyclic amorphous association schemes of three classes

Takuya Ikuta

Abstract

In this paper, we classify type II matrices and complex Hadamard matrices on pseudocyclic amorphous association schemes of class 3.

1 Introduction

Let X be a nonempty finite set with n elements. We denote by $M_X(\mathbb{C})$ the full matrix ring with complex entries whose rows and columns are indexed by X. Let $\mathbb{C}^*=\mathbb{C}-\{0\}$. Let $H\in M_X(\mathbb{C}^*)$. The (x, y)-entry of H is denoted by $H(x, y)$ for $\forall x, y\in X$. We assume that $|H(x, y)|=1$ for $\forall x, y\in X$. A matrix $H\in M_X(\mathbb{C}^*)$ is called a complex Hadamard matrix if the next equation holds:

For $\forall a, b\in X$

$$H\overline{H^T}=nI \Longleftrightarrow \sum_{x\in X} H(a, x)\overline{H(b, x)}=n\delta_{a, b}$$

$$\Longleftrightarrow \sum_{x\in X} \frac{H(a, x)}{H(b, x)}=n\delta_{a, b}. \quad (1)$$

The following is one of complex Hadamard matrices:

(1) a Hdamard matirx,

(2) the character table of a finite abelian group.

A matrix $W \in M_X(\mathbb{C}^*)$ is called a type II matrix if the next equation holds:

$$\sum_{x \in X} \frac{W(a, x)}{W(b, x)} = n\delta_{a,b} \qquad \forall a, b \in X \tag{2}$$

From (1), (2), we know that a complex Hadamard matrix H is one of type II matrices.

We consider complex Hadamard matrices within the framework of association schemes. Let $H \in M_X(\mathbb{C}^*)$ be a complex Hadamard matrix. Let $\chi = (X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme of class d. Let $\{A_i\}_{i=0}^d$ be the set of adjacency matrices for $\{R_i\}_{i=0}^d$. Let $\langle A_0 = I, A_1, \ldots, A_d \rangle$ be the Bose-Mesner algebra for $\chi = (X, \{R_i\}_{i=0}^d)$. Then, we consider the next expression:

$$H = \sum_{i=0}^{d} x_i A_i, \tag{3}$$

where $x_i \in \mathbb{C}^*$ and $|x_i| = 1$ for $\forall i = 0, \ldots, d$.

The aim of our recent research is to construct and classify complex Hadamard matrices satisfying the equation (3). Recently, we have constructed two new infinite series of complex Hadamard matrices on some association schemes. It seems to be difficult to classify complex Hadamard matrices within the framework of association schemes. We take notice some special association schemes, namely, pseudocyclic amorphous association schemes $\chi = (X, \{R_0, R_1, R_2, R_3\})$ of class 3, and classify complex Hadamard matrices attached to such association schemes. The first eigenmatrix P of pseudocyclic amorphous association schemes is given by

$$P=\begin{bmatrix} 1 & \frac{1}{3}(q^2-1) & \frac{1}{3}(q^2-1) & \frac{1}{3}(q^2-1) \\ 1 & -\frac{1}{3}(2q+1) & \frac{1}{3}(q-1) & \frac{1}{3}(q-1) \\ 1 & \frac{1}{3}(q-1) & -\frac{1}{3}(2q+1) & \frac{1}{3}(q-1) \\ 1 & \frac{1}{3}(q-1) & \frac{1}{3}(q-1) & -\frac{1}{3}(2q+1) \end{bmatrix}, \qquad (4)$$

where q is a positive integer, and $n=q^2$. In what follows, we assume that $q\geq 3$.

Then we have the following:

Theorem 1. *Let $H\in M_X(\mathbb{C}^*)$ be a type II matrix. Let $\chi=(X, \{R_0, R_1, R_2, R_3\})$ be a pseudocyclic amorphous association scheme of class 3. We write*

$$H=\sum_{i=0}^{3} x_i A_i.$$

where $x_i\in\mathbb{C}^*$ and $x_0=1$. We set

$$X_i=x_i+\frac{1}{x},$$

$$X_{ij}=\frac{x_i}{x_j}+\frac{x_j}{x_i}$$

for $1\leq i<j\leq 3$.

Then X_1, X_2, X_3, X_{12}, X_{13}, X_{23} are one of the following:

(1) $X_1=X_2=X_3=-q^2+2$, $X_{12}=X_{13}=X_{23}=2$,

(2) $X_1=X_{23}=-2$, $X_2=-X_3=-X_{12}=X_{13}$,

(3) $X_1=-2$, $X_2=X_3=\frac{q-4}{q-1}$, $X_{12}=X_{13}=\frac{-q+4}{q-1}$, $X_{23}=\frac{-q^2-4q+14}{(q-1)^2}$,

(4) $X_1=\frac{q^2-2q-2}{q-1}$, $X_2=X_3=\frac{-q^2+2q-4}{2(q-1)}$,

$$X_{12}=X_{13}=\frac{-5q^2+10q+4}{2(q-1)^2},\ X_{23}=2,$$

(5) $X_1=2,\ X_2=X_3=X_{12}=X_{13}=\dfrac{-q-2}{q-1},\ X_{23}=\dfrac{-q^2+8q+2}{2(q-1)^2}$,

(6) $X_i=X_{jk}=2$ *for* $\{i, j, k\}=\{1, 2, 3\}$, *the others* $X_{i_1}=X_{i_1i_2}=-2$,

(7) $X_{ij}=2,\ X_i=X_j=-q+2$ *for* $i, j\in\{1, 2, 3\}$, $X_k=\dfrac{q^2+2}{2q+1}$,

$$X_{ki}=X_{kj}=\frac{-5q+2}{2q+1}\ \textit{for}\ k=\{1, 2, 3\}-\{i, j\},$$

(8) *Let*

$$X_1=-q+2,$$

$$X_2=-\frac{(q-1)^2X_{12}+(q-4)(2q+1)}{3(q-1)},$$

$$X_3=\frac{(q-1)^2X_{12}+q^2-2q-8}{3(q-1)},$$

$$X_{13}=-\frac{(q-1)X_{12}-q+4}{q-1},$$

$$X_{23}=-\frac{4q^2-11q-2}{(q-1)^2}.$$

We set

$$a=(2q+1)(q-1)^2,$$

$$b=-(q-1)(2q+1)(q-4),$$

$$c=-13q^3+30q^2+6q+4.$$

Then, X_{12} *satisfies the next equation:*

$$aX_{12}^2+bX_{12}+c=0.$$

(9) *Let*

$$X_1=-\frac{1}{3}(q-1)(X_{12}+X_{13}+2),$$

$$X_2=\frac{(q-1)^2X_{13}+q^2-2q-8}{3(q-1)},$$

$$X_3=\frac{(q-1)^2X_{12}+q^2-2q-8}{3(q-1)},$$

$$X_{23}=-\frac{(q-1)^2(X_{12}+X_{13})+3(q^2-2q-2)}{(q-1)^2}.$$

We set

$a=(q-1)^4(X_{12}+2)$,

$b=(q-1)^2(X_{12}+2)((q-1)^2X_{12}+3q^2-6q-6)$,

$c=2(q-1)^4X_{12}^2+6(q^2-2q-2)(q-1)^2X_{12}$
$\quad+(q+2)(q-4)(5q^2-10q-4)$.

Then, X_{12}, X_{13} *satisfy the next equation:*

$$aX_{13}^2+bX_{13}+c=0.$$

Theorem 2. *Let* $H\in M_X(\mathbb{C}^*)$ *be a complex Hadamard matrix. Let* $\chi=(X, \{R_0, R_1, R_2, R_3\})$ *be a pseudocyclic amorphous association scheme of class* 3. *We write*

$$H=\sum_{i=0}^{3}x_iA_i.$$

where $x_i\in\mathbb{C}^*$, $|x_i|=1$ *for* $i=0, 1, 2, 3$ *and* $x_0=1$. *We set*

$$X_i=x_i+\frac{1}{x},$$

$$X_{ij}=\frac{x_i}{x_j}+\frac{x_j}{x_i}$$

for $1\leq i<j\leq 3$.

Then, one of the following holds

(1) $X_i=X_{jk}=2$ *for* $\{i, j, k\}=\{1, 2, 3\}$, *the others* $X_{i_1}=X_{i_1i_2}=-2$ *(a Hadamard matrix)*,

(2) $X_1=X_{23}=-2$, $X_2=-X_3=-X_{12}=X_{13}$,

(3) $X_1=-2$, $X_2=X_3=\dfrac{q-4}{q-1}$, $X_{12}=X_{13}=\dfrac{-q+4}{q-1}$, $X_{23}=\dfrac{-q^2-4q+14}{(q-1)^2}$,

(4) $X_1=2,\ X_2=X_3=X_{12}=X_{13}=\dfrac{-q-2}{q-1},\ X_{23}=\dfrac{-q^2+8q+2}{2(q-1)^2},$

(5) $X_1=-1,\ X_2=-\dfrac{2}{3}X_{12}-\dfrac{7}{6},\ X_3=\dfrac{2}{3}X_{12}-\dfrac{5}{6},\ X_{13}=-X_{12}-\dfrac{1}{2},$

$X_{23}=-\dfrac{1}{4}$, *where* X_{12} *is a solution of*

$$28X_{12}^2+14X_{12}-59=0:$$

(6) *Let*

$$X_1=-\frac{1}{3}(q-1)(X_{12}+X_{13}+2),$$

$$X_2=\frac{(q-1)^2X_{13}+q^2-2q-8}{3(q-1)},$$

$$X_3=\frac{(q-1)^2X_{12}+q^2-2q-8}{3(q-1)},$$

$$X_{23}=-\frac{(q-1)^2(X_{12}+X_{13})+3(q^2-2q-2)}{(q-1)^2}.$$

We set

$a=(q-1)^4(X_{12}+2),$

$b=(q-1)^2(X_{12}+2)((q-1)^2X_{12}+3q^2-6q-6),$

$c=2(q-1)^4X_{12}^2+6(q^2-2q-2)(q-1)^2X_{12}$
$\quad+(q+2)(q-4)(5q^2-10q-4).$

Then, X_{12}, X_{13} *satisfy the next equation:*

$$aX_{13}^2+bX_{13}+c=0.$$

2　Complex Hadamard matrices, Pseudocyclic amor-phous association schemes of class 3

In this section, we introduce the known results.

Let P be the first eigenmatrix of pseudocyclic amorphous association schemes of class 3 as the following:

$$P=\begin{bmatrix} 1 & \frac{1}{3}(q^2-1) & \frac{1}{3}(q^2-1) & \frac{1}{3}(q^2-1) \\ 1 & -\frac{1}{3}(2q+1) & \frac{1}{3}(q-1) & \frac{1}{3}(q-1) \\ 1 & \frac{1}{3}(q-1) & -\frac{1}{3}(2q+1) & \frac{1}{3}(q-1) \\ 1 & \frac{1}{3}(q-1) & \frac{1}{3}(q-1) & -\frac{1}{3}(2q+1) \end{bmatrix}, \quad (5)$$

where q is a positive integer, and $n=q^2$.

```
F<q>:=FunctionField(Rationals());
r:=(q-1)/3;
s:=r;
t:=-(1+r+s);
k:=-(r*s+s*t+t*r);
n:=3*k+1;
n eq q^2;
P:=Matrix(F,4,4,[
  1,k,k,k,
  1,t,s,r,
  1,r,t,s,
  1,s,r,t
]);
/*
[        1  1/3*q^2 -1/3 1/3*q^2 -1/3 1/3*q^2 -1/3]
[        1  -2/3*q -1/3   1/3*q -1/3    1/3*q -1/3]
[        1    1/3*q -1/3 -2/3*q -1/3    1/3*q -1/3]
[        1    1/3*q -1/3   1/3*q -1/3  -2/3*q -1/3]
*/
```

The next theorem is very useful to find complex Hadamard matrices.

Theorem 3. *Let X_1, X_2, X_3, X_{12}, X_{13}, X_{23} be real numbers satisfying*

$$g_{12}=X_1^2-X_1X_2X_{12}+X_2^2+X_{12}^2-4=0, \tag{6}$$

$$g_{13}=X_1^2-X_1X_3X_{13}+X_3^2+X_{13}^2-4=0, \tag{7}$$

$$g_{23}=X_2^2-X_2X_3X_{23}+X_3^2+X_{23}^2-4=0, \tag{8}$$

$$g_{24}=X_{12}^2-X_{12}X_{13}X_{23}+X_{13}^2+X_{23}^2-4=0, \tag{9}$$

$$h_{12}=2X_1X_2-X_1X_3X_{23}-X_2X_3X_{13}+X_3^2X_{12}-4X_{12} +2X_{13}X_{23}=0, \tag{10}$$

$$h_{13}=-X_1X_2X_{23}+2X_1X_3+X_2^2X_{13}-X_2X_3X_{12} +2X_{12}X_{23}-4X_{13}=0, \tag{11}$$

$$h_{23}=X_1^2X_{23}-X_1X_2X_{13}-X_1X_3X_{12}+2X_2X_3+2X_{12}X_{13}-4X_{23}=0. \tag{12}$$

and assume $X_1^2\neq 4$. Let x_1 be a complex number satisfying

$$x_1^2-X_1x_1+1=0. \tag{13}$$

Define complex numbers x_2, x_3 by

$$x_2=\frac{X_1x_1-2}{X_{12}x_1-X_2}, \tag{14}$$

$$x_3=\frac{X_1x_1-2}{X_{13}x_1-X_3}, \tag{15}$$

Then

$$\frac{x_j}{x_i}+\frac{x_i}{x_j}=X_{ij}\quad (1\leq i<j\leq 3),\quad x_i+\frac{1}{x_i}=X_i\quad (i=1,2,3). \tag{16}$$

Moreover, if $-2<X_1<2$, then $|x_1|=|x_2|=|x_3|=1$.

Proof. First we need to check the denominators of (14) and (15) are nonzero.

If $X_{12}x_1-X_2=0$, then by (13), we have

$$\frac{X_2^2}{X_{12}^2}-X_1\frac{X_2}{X_{12}}+1=0,$$

or equivalently,

$$X_2^2-X_1X_2X_{12}+X_{12}^2=0.$$

Together with (20), this implies $X_1^2=4$, which is a contradiction. Therefore, x_2 is well-de.ned.

Similarly, if $X_{13}x_1-X_3=0$, then we obtain $X_1^2=4$ using (21), which is a contradiction. Therefore, x_3 is well-de.ned.

```
/* Now we can check that (16) is satis.ed.
*/
Q:=Rationals();
R7<x1,X1,X2,X3,X12,X13,X23>:=PolynomialRing(Q,7);
xi1:=x1^2-X1*x1+1;
g:=X12^2+X13^2+X23^2-X12*X13*X23-4;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
gen:=[xi1,g,g12,g13,g23,h12,h13,h23];
I:=ideal<R7|gen>;
F7<y1,Y1,Y2,Y3,Y12,Y13,Y23>:=FieldOfFractions(R7);
y2:=(Y1*y1-2)/(Y12*y1-Y2);
y3:=(Y1*y1-2)/(Y13*y1-Y3);
{Numerator(y1+1/y1-Y1), Numerator(y2+1/y2-Y2),
  Numerator(y3+1/y3-Y3),
  Numerator(y1/y2+y2/y1-Y12), Numerator(y1/y3+y3/y1
    -Y13),
```

```
  Numerator(y2/y3+y3/y2-Y23)} subset I;
/*
```

Moreover, if $-2<X_1<2$, then by (13), x_1 is an imaginary number with $|x_1|=1$. In this case, we can check $|x_2|=|x_3|=1$ as follows.

```
*/
y2bar:=(y1^(-2)-1)/(Y12*y1^(-1)-Y2);
y3bar:=(y1^(-2)-1)/(Y13*y1^(-1)-Y3);
// y3bar:=(y1^(-2)-y2^(-2))^(-1)*(y1^(-2)*y2^(-1)*
           Y23-y1^(-1)*y2^(-2)*Y13);
{Numerator(y2*y2bar-1), Numerator(y3*y3bar-1)} subset
  I;
/*
```

□

We find complex Hadamard matrices on pseudocyclic amorphous association of three classes. To do this is to check the next Lemma:

Lemma 1. *Let* $H\in M_X(\mathbb{C}^*)$. *Let* $\chi=(X, \{R_0, R_1, R_2, R_3\})$ *be a pseudocyclic amorphous association scheme of three classes, and* P *be the first eigenmatrix given by* (5). *Let*

$$H=\sum_{i=0}^{3} x_i A_i.$$

We set

$$X_i=x_i+\frac{1}{x_i},$$

$$X_{ij}=\frac{x_i}{x_j}+\frac{x_j}{x_i}$$

for $1\leq i<j\leq 3$. *Then, if the next seven equations hold, then* H *is a type II matrix:*

$$e_1=-3(2q+1)X_1+3(q-1)(X_2+X_3)+(q^2-2q+1)X_{23}$$

$$+(-2q^2+q+1)(X_{12}+X_{13})+3(-q^2+4)=0, \tag{17}$$

$$\begin{aligned} e_2=&-3(2q+1)X_2+3(q-1)(X_1+X_3)\\ &+(-2q^2+q+1)(X_{12}+X_{13})+(q^2-2q+1)X_{13}\\ &+3(-q^2+4)=0, \end{aligned} \tag{18}$$

$$\begin{aligned} e_3=&-3(2q+1)X_3+3(q-1)(X_1+X_2)+(q^2-2q+1)X_{12}\\ &+(-2q^2+q+1)(X_{13}+X_{23})+3(-q^2+4)=0, \end{aligned} \tag{19}$$

$$g_{12}=X_1^2-X_1X_2X_{12}+X_2^2+X_{12}^2-4=0, \tag{20}$$

$$g_{13}=X_1^2-X_1X_3X_{13}+X_3^2+X_{13}^2-4=0, \tag{21}$$

$$g_{23}=X_2^2-X_2X_3X_{23}+X_3^2+X_{23}^2-4=0, \tag{22}$$

$$g_{24}=X_{12}^2-X_{12}X_{13}X_{23}+X_{13}^2+X_{23}^2-4=0, \tag{23}$$

$$h_{12}=2X_1X_2-X_1X_3X_{23}-X_2X_3X_{13}+X_3^2X_{12}-4X_{12}+2X_{13}X_{23}=0, \tag{24}$$

$$\begin{aligned} h_{13}=&-X_1X_2X_{23}+2X_1X_3+X_2^2X_{13}-X_2X_3X_{12}\\ &+2X_{12}X_{23}-4X_{13}=0, \end{aligned} \tag{25}$$

$$h_{23}=X_1^2X_{23}-X_1X_2X_{13}-X_1X_3X_{12}+2X_2X_3+2X_{12}X_{13}-4X_{23}=0. \tag{26}$$

Proof.

```
P6<X1,X2,X3,X12,X13,X23>:
        =PolynomialRing(F,6);
 e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
     +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
     +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
 e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
     +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
     +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
 e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
     +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
     +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
```

```
g23:＝X23^2－X2*X3*X23＋X2^2＋X3^2－4;
g24:＝X12^2＋X13^2＋X23^2－X12*X13*X23－4;
h12:＝(X3^2－4)*X12－X3*(X1*X23＋X2*X13)
      ＋2*(X1*X2＋X13*X23);
h13:＝(X2^2-4)*X13－X2*(X1*X23＋X3*X12)
      ＋2*(X1*X3＋X12*X23);
h23:＝(X1^2－4)*X23－X1*(X2*X13＋X3*X12)
      ＋2*(X2*X3＋X12*X13);
eq10:＝[e1,e2,e3,g12,g13,g23,g24,h12,h13,h23];
/*
[
    (－2/3*q－1/3)*X1＋(1/3*q－1/3)*X2＋(1/3*q－1/3)*X3
      ＋(－2/9*q^2＋1/9*q＋1/9)*X12＋(－2/9*q^2＋1/9*q
      ＋1/9)*X13＋(1/9*q^2－2/9*q＋1/9)*X23－1/3*q^2
      ＋4/3,
    (1/3*q－1/3)*X1＋(－2/3*q－1/3)*X2＋(1/3*q－1/3)*X3
      ＋(－2/9*q^2＋1/9*q＋1/9)*X12＋(1/9*q^2－2/9*q
      ＋1/9)*X13＋(－2/9*q^2＋1/9*q＋1/9)*X23－1/3*q^2
      ＋4/3,
    (1/3*q－1/3)*X1＋(1/3*q－1/3)*X2＋(－2/3*q－1/3)*X3
      ＋(1/9*q^2－2/9*q＋1/9)*X12＋(－2/9*q^2＋1/9*q
      ＋1/9)*X13＋(－2/9*q^2＋1/9*q＋1/9)*X23－1/3*q^2
      ＋4/3,
    X1^2－X1*X2*X12＋X2^2＋X12^2－4,
    X1^2－X1*X3*X13＋X3^2＋X13^2－4,
    X2^2－X2*X3*X23＋X3^2＋X23^2－4,
    X12^2－X12*X13*X23＋X13^2＋X23^2－4,
    2*X1*X2－X1*X3*X23－X2*X3*X13＋X3^2*X12－4*X12
```

```
  +2*X13*X23,
 -X1*X2*X23+2*X1*X3+X2^2*X13-X2*X3*X12
  +2*X12*X23-4*X13,
 X1^2*X23-X1*X2*X13-X1*X3*X12+2*X2*X3+2*X12*X13
  -4*X23
]
*/
```

□

3 Proof of Theorem 1

Lemma 2. *X_2, X_3 are given by the following:*

$$X_2 = X_1 + \frac{q-1}{3}(X_{13} - X_{23}),$$

$$X_3 = X_1 + \frac{q-1}{3}(X_{12} - X_{23}).$$

Proof. From e1−e2, e1−e3 we have the following:

```
e1-e2 eq-(1/3)*q*(X13*q-X23*q+3*X1-3*X2-X13+X23);
e1-e3 eq-(1/3)*q*(X12*q-X23*q+3*X1-3*X3-X12+X23);
// From e1-e2, we have
//    X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q
          +(1/3)*X23;
// From e1-e3, we have
//    X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q
          +(1/3)*X23;
```

□

Lemma 3. *Under Lemma 2, X_1 is given by the following:*

$$X_1 = -\frac{1}{9}(q+2)(q-1)(X_{12}+X_{13}) - \frac{1}{9}(q-1)^2 X_{23} - \frac{1}{3}(q-2)(q+2).$$

Proof.
```
P4<X1,X12,X13,X23>:=PolynomialRing(F,4);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
h1:=hom<P6->P4|[X1,X2,X3,X12,X13,X23]>;
e11:=e1 @ h1;
e21:=e2 @ h1;
e31:=e3 @ h1;
e11 eq e21;
e11 eq e31;
e11 eq-(1/9)*X13*q^2-(1/9)*X13*q-(1/9)*X23*q^2
    +(2/9)*X23*q-(1/9)*X12*q^2-(1/9)*X12*q
    -(1/3)*q^2+4/3-X1+(2/9)*X12+(2/9)*X13-(1/9)*X23;
// From this eqaution, we have
// X1:=-(1/3)*q^2+(2/9)*X23*q-(1/9)*X13*q^2
    -(1/9)*X13*q-(1/9)*X23*q^2
// -(1/9)*X23+(2/9)*X13-(1/9)*X12*q^2-(1/9)*X12*q
    +(2/9)*X12+4/3;
```
□

Using Lemmas 2, 3, e1=e2=e3=0 hold in Thorem 3. Therefore, it is sufficient to find X_{12}, X_{13}, X_{23} which (20)−(26) hold.

In what follows, X_1, X_2, X_3 are given in Lemmas 2, 3. Then we have the following:

Lemma 4. *Let X_1, X_2, X_3 be given in Lemmas 2,3. Then the next equation with respect to X_{12}, X_{13}, X_{23} holds: Under Lemmas 2, 3, we have the following:*

$$(X_{12}-X_{13})((q+2)(X_{12}+X_{13})+(q-1)X_{23}+3q-6)$$
$$\cdot((q-1)^2(X_{12}+X_{13}+X_{23})+3(q^2-2q-2))=0.$$

Proof.
```
P3<X12,X13,X23>:=PolynomialRing(F,3);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
    -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
       +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
       +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
       +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
       +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
       +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
       +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
g12-g13 eq -(1/81*(q+2))*(X12-X13)*((q+2)*X12
      +(q+2)*X13+(q-1)*X23+3*q-6)
*((q-1)^2*(X12+X13+X23)+3*(q^2-2*q-2));
```

Since $q \neq -2 (n \neq 4)$, we have the assertion. □

From Lemma 4 we consider the next three cases:

$$\begin{cases} \text{Case A:} & X_{12}-X_{13}=0, \\ \text{Case B:} & (q+2)(X_{12}+X_{13})+(q-1)X_{23}+3q-6=0, \\ \text{Case C:} & (q-1)^2(X_{12}+X_{13}+X_{23})+3(q^2-2q-2)=0. \end{cases}$$

3.1 Case A: $X_{12}-X_{13}=0$.

In this subsection, we assume that $X_{13}=X_{12}$. Then we have the following:

Lemma 5. *Let $X_{13}=X_{12}$. Then the next equation holds:*

$$(X_{12}-X_{23})((2q+1)X_{12}+(q+2)X_{23}+3q-6) \cdot(2(q-1)^2X_{12}+(q-1)^2X_{23}+3q^2-6q-6)=0. \quad (27)$$

Proof.
```
P2<X12,X23>:=PolynomialRing(F,2);
X13:=X12;
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
    -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
       +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
       +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
       +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
       +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
       +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
       +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
```

```
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
       +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
       +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
       +2*(X2*X3+X12*X13);
// Then we have the following:
g12-g23 eq -(1/81*(q+2))*(X12-X23)*((2*q+1)*X12
    +(q+2)*X23+3*q-6)
*(2*(q-1)^2*X12+(q-1)^2*X23+3*q^2-6*q-6);
```

Since we asumme $q \neq -2$, we have the assertion. □

From (27), we consider the next three cases:

$$\begin{cases} \text{Case A-1:} & X_{12}-X_{23}=0, \\ \text{Case A-2:} & (2q+1)X_{12}+(q+2)X_{23}+3q-6=0, \\ \text{Case A-3:} & 2(q-1)^2X_{12}+(q-1)^2X_{23}+3q^2-6q-6=0. \end{cases}$$

3.1.1 Case A-1: $X_{12}-X_{23}=0$.

Let $X_{23}=X_{12}$. Then we have the following:

Lemma 6. *Let $X_{23}=X_{13}=X_{12}$. Then we have the following:*

$$(X_{12}+1)(X_{12}-2)((q^2+2)X_{12}-4+q^2)=0. \quad (28)$$

Proof.
```
P1<X12>:=PolynomialRing(F);
X13:=X12;
X23:=X12;
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
      -1/3*(q-2)*(q+2);
```

```
X2:＝X1＋(1/3)*X13*q－(1/3)*X13－(1/3)*X23*q＋(1/3)*X23;
X3:＝X1＋(1/3)*X12*q－(1/3)*X12－(1/3)*X23*q＋(1/3)*X23;
  e1:＝1＋P[2,2]^2＋P[2,3]^2＋P[2,4]^2＋P[2,2]*X1
       ＋P[2,3]*X2＋P[2,4]*X3＋P[2,2]*P[2,3]*X12
       ＋P[2,2]*P[2,4]*X13＋P[2,3]*P[2,4]*X23－n;
  e2:＝1＋P[3,2]^2＋P[3,3]^2＋P[3,4]^2＋P[3,2]*X1
       ＋P[3,3]*X2＋P[3,4]*X3＋P[3,2]*P[3,3]*X12
       ＋P[3,2]*P[3,4]*X13＋P[3,3]*P[3,4]*X23－n;
  e3:＝1＋P[4,2]^2＋P[4,3]^2＋P[4,4]^2＋P[4,2]*X1
       ＋P[4,3]*X2＋P[4,4]*X3＋P[4,2]*P[4,3]*X12
       ＋P[4,2]*P[4,4]*X13＋P[4,3]*P[4,4]*X23－n;
g12:＝X12^2－X1*X2*X12＋X1^2＋X2^2－4;
g13:＝X13^2－X1*X3*X13＋X1^2＋X3^2－4;
g23:＝X23^2－X2*X3*X23＋X2^2＋X3^2－4;
g24:＝X12^2＋X13^2＋X23^2－X12*X13*X23－4;
h12:＝(X3^2－4)*X12－X3*(X1*X23＋X2*X13)
       ＋2*(X1*X2＋X13*X23);
h13:＝(X2^2－4)*X13－X2*(X1*X23＋X3*X12)
       ＋2*(X1*X3＋X12*X23);
h23:＝(X1^2－4)*X23－X1*(X2*X13＋X3*X12)
       ＋2*(X2*X3＋X12*X13);
g12－g24 eq －(1/9*(q－2))*(q＋2)*(X12＋1)*(X12－2)*
       ×((q^2＋2)*X12－4＋q^2);
```

□

From (28) we consider the next three cases:

$$\begin{cases} \text{Case A-1-1:} & X_{12}+1=0, \\ \text{Case A-1-2:} & X_{12}-2=0, \\ \text{Case A-1-3:} & (q^2+2)X_{12}-4+q^2=0. \end{cases}$$

For the above cases, we have the following:

Lemma 7. *Let $X_{23}=X_{13}=X_{12}$. Then, Cases A-1-1, A-1-2 does not satisfy a type II matrix. CaseA-1-2 is reduced to the following:*

$$X_1=X_2=X_3=-q^2+2,\ X_{12}=X_{13}=X_{23}=2.$$

Proof.
```
////CaseA-1-1 : X12+1=0:
CaseA11:=hom<P1->F|[-1]>;
h12 @ CaseA11 eq 9;
// This is a contradiction.

//// CaseA-1-2:X12-2=0:
CaseA12:=hom<P1->F|[2]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
&and[e @ CaseA12 eq 0 : e in eq7];
X1 @ CaseA12 eq-q^2+2;
X2 @ CaseA12 eq-q^2+2;
X3 @ CaseA12 eq-q^2+2;
X12 @ CaseA12 eq 2;
X13 @ CaseA12 eq 2;
X23 @ CaseA12 eq 2;
// X1=X2=X3=-q^2+2, X12=X13=X23=2.

//// CaseA-1-3 : (q^2+2)*X12-4+q^2=0 :
CaseA13:=hom<P1->F|[-(-4+q^2)/(q^2+2)]>;
g12 @ CaseA13 eq -54*q^4/(q^6 + 6*q^4 + 12*q^2 + 8);
```

```
// This is a contradiction.
```

□

3.1.2 Case A-2: $(2q+1)X_{12}+(q+2)X_{23}+3q-6=0$.

Let $X_{23}=-\dfrac{(2q+1)X_{12}+3q-6}{q+2}$. Then we have the following:

Lemma 8. *Let* $X_{13}=X_{12}$, $X_{23}=-\dfrac{(2q+1)X_{12}+3q-6}{q+2}$. *Then we have the following:*

$$((2q+1)X_{12}+5q-2)(X_{12}+2-q)(X_{12}-2+q)=0. \quad (29)$$

Proof.
```
P1<X12>:=PolynomialRing(F);
X13:=X12;
X23:=-(3*q+2*X12*q-6+X12)/(q+2);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
      -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
        +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
        +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
        +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
        +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
        +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
        +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
```

```
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
     +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
     +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
     +2*(X2*X3+X12*X13);
g12-g24 eq -((2*q+1)*X12+5*q-2)*(X12+2-q)*
     ×(X12-2+q)/(q+2);
```

□

From (29) we have the next three cases:

$$\begin{cases} \text{Case A-2-1:} \quad (2q+1)X_{12}+5q-2=0, \\ \text{Case A-2-2:} \quad X_{12}+2-q=0, \\ \text{Case A-2-3:} \quad X_{12}-2+q=0. \end{cases}$$

For the above cases, we have the following:

Lemma 9. *Let* $X_{13}=X_{12}$, $X_{23}=-\dfrac{(2q+1)X_{12}+3q-6}{(q+2)}$. *Then, Case A-2-2 does not satisfy a type II matrix. Cases A-2-1, A-2-3 are reduced to the following, respectively:*

(1) $X_1=\dfrac{q^2+2}{2q+1}$, $X_2=X_3=-q+2$, $X_{12}=X_{13}=\dfrac{-5q+2}{2q+1}$, $X_{23}=2$,

(2) $X_1=2$, $X_2=-2$, $X_3=-2$, $X_{12}=-2$, $X_{13}=-2$, $X_{23}=2$.

Proof.
```
//// CaseA-2-1:(2*q+1)*X12+5*q-2=0:
CaseA21:=hom<P1->F|[-(5*q-2)/(1+2*q)]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
```

```
&and[e @ CaseA21 eq 0:e in eq7];
X1 @ CaseA21 eq (1/2*q^2+1)/(q+1/2);
X2 @ CaseA21 eq-q+2;
X3 @ CaseA21 eq-q+2;
X12 @ CaseA21 eq (-5/2*q+1)/(q+1/2);
X13 @ CaseA21 eq (-5/2*q+1)/(q+1/2);
X23 @ CaseA21 eq 2;
// X1=(1/2*q^2+1)/(q+1/2), X2=X3=-q+2, X12=X13
      =(-5/2*q+1)/(q+1/2), X23=2;

//// CaseA-2-2:X12+2-q=0:
CaseA22:=hom<P1->F|[-2+q]>;
g12 @ CaseA22 eq (2*(q-1))*(q^2-2*q-2);
g13 @ CaseA22 eq (2*(q-1))*(q^2-2*q-2);
g23 @ CaseA22 eq (2*(q-1))*(q^2-2*q-2);
g24 @ CaseA22 eq (2*(q-1))*(q^2-2*q-2);
h12 @ Case A22 eq (2*(q-1))*(-2+q)*(q^2-2*q-2);
h13 @ Case A22 eq (2*(q-1))*(-2+q)*(q^2-2*q-2);
h23 @ Case A22 eq -2*q*(q-1)*(-2+q)*(q^2-2*q-2);
// Discriminat of q^2-2*q-2 is minus.

//// CaseA-2-3:X12-2+q=0:
CaseA23:=hom<P1->F|[-q+2]>;
g12 @ CaseA23 eq -2*q^3*(q-4)^2/(q+2)^2;
// From this we have q=4.
CaseA232:=hom<P1->F|[-2]>;
g12 @ CaseA232;
g13 @ CaseA232;
```

```
g23 @ CaseA232;
g24 @ CaseA232;
h12 @ CaseA232;
h13 @ CaseA232;
h23 @ CaseA232;
caseA23:=[e @ Case A232:e in eq7];case A23;
/*
[
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^2-8*q+16)/(q^2+4*q+4),
    (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
    (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
    (2*q^5-8*q^4-27*q^3+88*q^2+80*q)/
        (q^3+6*q^2+12*q+8)
]
*/
// The next is due to Maple:
/*
A:=
[
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^2-8*q+16)/(q^2+4*q+4),
    (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
    (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
```

```
    (2*q^5-8*q^4-27*q^3+88*q^2+80*q)/
        (q^3+6*q^2+12*q+8)

];
q:=4; A;
*/
kai:=[X1,X2,X3,X12,X13,X23];
kaiA23:=[e @ Case A232:e in kai];kaiA23;
/*
[
    3*q/(q+2),
    -q+2,
    -q+2,
    -2,
    -2,
    (q+8)/(q+2)
]
*/
// X1:=2, X2:=-2, X3:=-2, X12:=-2, X13:=-2,
   X23:=2.
```

□

3.1.3 Case A-3: $2(q-1)^2X_{12}+(q-1)^2X_{23}+3q^2-6q-6=0.$

Let $X_{23}=-\dfrac{2(q-1)^2X_{12}+3q^2-6q-6}{(q-1)^2}$. Then we have the following:

Lemma 10. *Let* $X_{13}=X_{12}$, $X_{23}=-\dfrac{2(q-1)^2X_{12}+3q^2-6q-6}{(q-1)^2}$. *Then we have the following:*

$$(q-4)((q-1)X_{12}-4+q)(2(q-1)^2X_{12}+5q^2-10q-4)$$
$$\times((q-1)X_{12}+2+q)=0. \quad (30)$$

Proof.
```
P1<X12>:=PolynomialRing(F);
X13:=X12;
X23:=-(2*X12*q^2-4*X12*q+2*X12+3*q^2-6*q-6)/
     (q^2-2*q+1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
    -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
 e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
     +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
     +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
 e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
     +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
     +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
 e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
     +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
     +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
     +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
     +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
```

```
    +2*(X2*X3+X12*X13);
g12-g24 eq (1/9)*(q+2)*(q-4)*((q-1)*X12-4+q)
*(2*(q-1)^2*X12+5*q^2-10*q-4)*((q-1)*X12+2+q)/
    (q-1)^4;
```

□

From (30) we consider the next four cases:

$$\begin{cases} \text{Case A-3-1:} & q-4=0, \\ \text{Case A-3-2:} & (q-1)X_{12}-4+q=0, \\ \text{Case A-3-3:} & 2(q-1)^2X_{12}+5q^2-10q-4=0, \\ \text{Case A-3-4:} & (q-1)X_{12}+2+q=0. \end{cases}$$

For the above cases, we have the following:

Lemma 11. *Let* $X_{23}=X_{12}$, $X_{23}=-\dfrac{2(q-1)^2X_{12}+3q^2-6q-6}{(q-1)^2}$. *Then, Case A-3-1 does not satisfy a type II matrix. Cases A-3-2, A-3-3, A-3-4 are reduced to the following, respectively:*

(1) $X_1=-2$, $X_2=X_3=\dfrac{q-4}{q-1}$, $X_{12}=X_{13}=\dfrac{-q+4}{q-1}$, $X_{23}=\dfrac{-q-4q+14}{(q-1)^2}$,

(2) $X_1=\dfrac{q^2-2q-2}{q-1}$, $X_2=X_3=\dfrac{-q^2+2q-4}{2(q-1)}$,

$X_{12}=X_{13}=\dfrac{-5q^2+10q+4}{2(q-1)^2}$, $X_{23}=2$,

(3) $X_1=2$, $X_2=X_3=X_{12}=X_{13}=\dfrac{-q-2}{q-1}$, $X_{23}=\dfrac{-q^2+8q+2}{2(q-1)^2}$.

Proof. //// CaseA-3-1:q-4=0:

```
CaseA31:=hom<P1->F|[4]>;
g12 @ CaseA31 eq (1/9)*(5*q-8)*(5*q-2)*(13*q^2-26*q+4)
    /(q-1)^2;
```

```
// Since q is a positive, this is a contradiction.

//// CaseA-3-2:(q−1)*X12−4+q=0:
CaseA32:=hom<P1−>F|[−(q−4)/(q−1)]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
&and[e @ CaseA32 eq 0:e in eq7];
kai:=[X1,X2,X3,X12,X13,X23];
kaiA32:=[e @ Case A32:e in kai];
print "kaiA32";
kaiA32;
/*
[
    −2,
    (q−4)/(q−1),
    (q−4)/(q−1),
    (−q+4)/(q−1),
    (−q+4)/(q−1),
    (−q^2−4*q+14)/(q^2−2*q+1)
]
*/
// X1=−2, X2=X3=(q−4)/(q−1), X12=X13=(−q+4)/(q−1),
// X23=(−q^2−4*q+14)/(q^2−2*q+1).

//// CaseA-3-3: 2*X12*q^2−4*X12*q+2*X12+5*q^2−10*q
    −4=0:
CaseA33:=hom<P1−>F|[−(1/2)*(5*q^2−10*q−4)/
    (q^2−2*q+1)]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
```

```
&and[e @ CaseA33 eq 0:e in eq7];
kai:=[X1,X2,X3,X12,X13,X23];
kaiA33:=[e @ CaseA33:e in kai];
print "kaiA33;
kaiA33;
/*
[
    (q^2-2*q-2)/(q-1),
    (-1/2*q^2+q-2)/(q-1),
    (-1/2*q^2+q-2)/(q-1),
    (-5/2*q^2+5*q+2)/(q^2-2*q+1),
    (-5/2*q^2+5*q+2)/(q^2-2*q+1),
    2
]
*/
// X1=(q^2-2*q-2)/(q-1), X2=X3=(-1/2*q^2+q-2)/
    (q-1),
// X12=X13=(-5/2*q^2+5*q+2)/(q^2-2*q+1), X23=2.

//// CaseA-3-4 : X12*q-X12+2+q=0:
CaseA34:=hom<P1->F|[-(q+2)/(q-1)]>;
&and[e @ CaseA34 eq 0:e in eq7];
kai:=[X1,X2,X3,X12,X13,X23];
kaiA34:=[e @ Case A34:e in kai];
print "kaiA34";
kaiA34;
/*
[
```

```
    2,
    (-q-2)/(q-1),
    (-q-2)/(q-1),
    (-q-2)/(q-1),
    (-q-2)/(q-1),
    (-q^2+8*q+2)/(q^2-2*q+1)
]
*/
// X1=2, X2=X3=X12=X13=(-q-2)/(q-1), X23
     =(-q^2+8*q+2)/(q^2-2*q+1).
```

□

3.2 CaseB: $(q+2)(X_{12}+X_{13})+(q-1)X_{23}+3q-6=0.$

In this subsection, we assume that $X_{23}=-\dfrac{(q+2)(X_{12}+X_{13})+3q-6}{q-1}$. Then, we have the following:

Lemma 12. *Let* $X_{23}=-\dfrac{(q+2)(X_{12}+X_{13})+3q-6}{q-1}$. *Then we have the following:*

$$(X_{12}-2+q)(X_{12}+2-q)((2q+1)X_{12}-2+5q)=0. \qquad (31)$$

Proof.

```
P2<X12,X13>:=PolynomialRing(F,2);
X23:=-(X12*q+2*X12-6+2*X13+3*q+X13*q)/(q-1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
     -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
       +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
```

```
     ＋P[2,2]*P[2,4]*X13＋P[2,3]*P[2,4]*X23－n;
  e2:＝1＋P[3,2]^2＋P[3,3]^2＋P[3,4]^2＋P[3,2]*X1
     ＋P[3,3]*X2＋P[3,4]*X3＋P[3,2]*P[3,3]*X12
     ＋P[3,2]*P[3,4]*X13＋P[3,3]*P[3,4]*X23－n;
  e3:＝1＋P[4,2]^2＋P[4,3]^2＋P[4,4]^2＋P[4,2]*X1
     ＋P[4,3]*X2＋P[4,4]*X3＋P[4,2]*P[4,3]*X12
     ＋P[4,2]*P[4,4]*X13＋P[4,3]*P[4,4]*X23－n;
g12:＝X12^2－X1*X2*X12＋X1^2＋X2^2－4;
g13:＝X13^2－X1*X3*X13＋X1^2＋X3^2－4;
g23:＝X23^2－X2*X3*X23＋X2^2＋X3^2－4;
g24:＝X12^2＋X13^2＋X23^2－X12*X13*X23－4;
h12:＝(X3^2－4)*X12－X3*(X1*X23＋X2*X13)
      ＋2*(X1*X2＋X13*X23);
h13:＝(X2^2－4)*X13－X2*(X1*X23＋X3*X12)
      ＋2*(X1*X3＋X12*X23);
h23:＝(X1^2－4)*X23－X1*(X2*X13＋X3*X12)
      ＋2*(X2*X3＋X12*X13);
// Then we have the following:
g12－g23 eq －(1/9)*(q＋2)*((2*q＋1)*X12＋(q＋2)*X13
      ＋3*q－6)*
((q－1)*X12＋(q－1)*X13－q＋4)*((q＋2)*X12＋(2*q＋1)*X13
      ＋3*q－6)/(q－1)^2;
```

□

From (31) we have the next cases:

$$\begin{cases}\text{Case B-1:} & (2q+1)X_{12}+(q+2)X_{13}+3q-6=0,\\ \text{Case B-2:} & (q-1)X_{12}+(q-1)X_{13}-q+4=0,\\ \text{Case B-3:} & (q+2)X_{12}+(2q+1)X_{13}+3q-6=0\end{cases}$$

3.2.1 Case B-1: $(2q+1)X_{12}+(q+2)X_{13}+3q-6=0$.

Let $X_{13}=-\dfrac{(2q+1)X_{12}+3q-6}{q+2}$. Then we have the following:

Lemma 13.

Let $X_{23}=-\dfrac{(q+2)(X_{12}+X_{13})+3q-6}{q-1}$, $X_{13}=-\dfrac{(2q+1)X_{12}+3q-6}{q+2}$. *Then we have the following:*

$$(X_{12}-2+q)(X_{12}+2-q)((2q+1)X_{12}-2+5q)=0. \qquad (32)$$

Proof.
```
//// CaseB-1: (2*q+1)*X12+(q+2)*X13+3*q-6=0:
P1<X12>:=PolynomialRing(F);
X13:=-(2*X12*q+X12-6+3*q)/(q+2);
X23:=-(X12*q+2*X12-6+2*X13+3*q+X13*q)/(q-1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
     -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
       +P[2,3]*X2+P[2,4]*X3+ P[2,2]*P[2,3]*X12
       +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
       +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
       +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
       +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
       +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
```

```
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
g12-g24 eq -(X12-2+q)*(X12+2-q)*((2*q+1)*X12-2
      +5*q)/(q+2);
```

□

From (32) we consider the next three cases:

$$\begin{cases} \text{Case B-1-1:} \quad X_{12}-2+q=0, \\ \text{Case B-1-2:} \quad X_{12}+2-q=0, \\ \text{Case B-1-3:} \quad (2q+1)X_{12}-2+5q=0. \end{cases}$$

Lemma 14.

Let $X_{23}=-\dfrac{(q+2)(X_{12}+X_{13})+3q-6}{q-1}$, $X_{13}=-\dfrac{(2q+1)X_{12}+3q-6}{q+2}$. *Then Case B-1-2 does not satisfy a type II matrix. Cases B-1-1, B-1-3 are reduced to the following, respectively:*

(1) $X_1=X_3=X_{12}=X_{23}=-2$, $X_2=X_{13}=2$,

(2) $X_1=X_3=-q+2$, $X_2=\dfrac{q^2+2}{q+1}$, $X_{12}=X_{23}=\dfrac{-5q+1}{2q+1}$, $X_{13}=2$.

Proof. //// CaseB-1-1:X12-2+q=0:

```
CaseB11:=hom<P1->F|[-q+2]>;
g12 @ CaseB11 eq-2*(q-4)^2*q^3/(q+2)^2;
// From this we have q=4.
eq7:=[g12,g13,g23,g24,h12,h13,h23];
```

```
caseB11:=[e @ Case B11:e in eq7];caseB11;
/*
[
   (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
   (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
   (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
   (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
   (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4),
   (2*q^8-24*q^7+100*q^6-152*q^5+128*q^3)/
     (q^3+6*q^2+12*q+8),
   (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4)
]
*/
/*
By Maple,
A:=
[
   (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
   (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
   (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
   (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
   (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4),
   (2*q^8-24*q^7+100*q^6-152*q^5+128*q^3)/
     (q^3+6*q^2+12*q+8),
   (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4)
];
q:=4; A;
[0, 0, 0, 0, 0, 0, 0];
```

```
*/
kai:=[X1,X2,X3,X12,X13,X23];
kaiB11:=[e @ CaseB11:e in kai];
print "kaiB11";
kaiB11;
/*
[
    -q+2,
    (q^3-4*q^2+2*q+4)/(q+2),
    -q+2,
    -q+2,
    (2*q^2-6*q+4)/(q+2),
    -q+2
]
[-2, 2, -2, -2, 2, -2];
X1=X3=X12=X23=-2, X2=X13=2.
*/

//// CaseB-1-2:X12+2-q=0:
CaseB12:=hom<P1->F|[q-2]>;
g12 @ CaseB12 eq (2*(q-1))*(q^2-2*q-2);
// From this we have q=1.

//// CaseB-1-3:2*X12*q+X12-2+5*q=0:
Case B13:=hom<P1->F|[-(-2+5*q)/(2*q+1)]>;
&and[e @ Case B13 eq 0:e in eq7];
kai:=[X1,X2,X3,X12,X13,X23];
kaiB13:=[e @ CaseB13:e in kai];
```

```
print “kaiB13”;
kaiB13;
/*
[
    −q+2,
    (1/2*q^2+1)/(q+1/2),
    −q+2,
    (−5/2*q+1)/(q+1/2),
    2,
    (−5/2*q+1)/(q+1/2)
]
X1=X3=−q+2, X2=(1/2*q^2+1)/(q+1/2), X12=X23
  =(−5/2*q+1)/(q+1/2), X13=2.
*/
```

□

3.2.2 Case B-2: $(q-1)X_{12}+(q-1)X_{13}-q+4=0$.

Let $X_{13}=-\dfrac{(q-1)X_{12}-q+4}{q-1}$. Then we have the following:

Lemma 15.

Let $X_{23}=-\dfrac{(q+2)(X_{12}+X_{13})+3q-6}{q-1}$, $X_{13}=-\dfrac{(q-1)X_{12}-q+4}{q-1}$. *Then we have the following:*

$$(q-4)^2((2q+1)(q-1)^2X_{12}^2-(q-1)(2q+1)(q-4)X_{12}-13q^3+30q^2+6q+4)=0. \tag{33}$$

Proof.
```
//// Case B-2:(q−1)*X12+(q−1)*X13−q+4=0,
P1<X12>:=PolynomialRing(F);
X13:=−(X12*q−X12+4−q)/(q−1);
```

```
X23:=-(X12*q+2*X12-6+2*X13+3*q+X13*q)/(q-1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
     -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
      +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
      +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
      +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
      +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
      +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
      +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
g12-g24 eq -(1/9)*(q+2)*(q-4)^2
*((2*q+1)*(q-1)^2*X12^2-(q-1)*(2*q+1)*(q-4)*X12
      -13*q^3+30*q^2+6*q+4)/(q-1)^4;
```

□

From (33) we consider the next two cases:

$$\begin{cases} \text{Case B-2-1:} \quad q-4=0, \\ \text{Case B-2-2:} \quad (2q+1)(q-1)^2X_{12}^2-(q-1)(2q+1)(q-4)X_{12} \\ \qquad\qquad -13q^3+30q^2+6q+4=0. \end{cases}$$

For these, we have the following:

Lemma 16.

Let $X_{23}=-\dfrac{(q+2)(X_{12}+X_{13})+3q-6}{q-1}$, $X_{13}=-\dfrac{(q-1)X_{12}-q+4}{q-1}$.

(1) $X_1=-2$, $X_2=-X_{12}$, $X_3=X_{12}$, $X_{13}=-X_{12}$, $X_{23}=-2$,

(2) *We set*

$$a=(2q+1)(q-1)^2,$$
$$b=-(q-1)(2q+1)(q-4),$$
$$c=-13q^3+30q^2+6q+4,$$
$$p=aX_{12}^2+bX_{12}+c.$$

If $p=0$, then there exists a type II matrices.

Proof. print "general:";

```
//// In general:
g12 eq g13;
g12 eq g23;
h12 eq h13;
h12 eq h23;
pp:=(2*q+1)*(q-1)^2*X12^2-(q-1)*(2*q+1)*(q-4)*X12
    -13*q^3+30*q^2+6*q+4;
g12 eq -(1/9)*(q-4)*pp/(q-1)^2;
g24 eq -(q-4)*pp/(q-1)^4;
h12 eq (2/9)*(q-4)*pp/(q-1)^2;
```

```
kai:=[X1,X2,X3,X12,X13,X23];
kai;
/*
[
    -q + 2,
    (-1/3*q+1/3)*X12+(2/3*q^2-7/3*q-4/3)/(q-1),
    (1/3*q-1/3)*X12+(1/3*q^2-2/3*q-8/3)/(q-1),
    X12,
    -X12+(q-4)/(q-1),
    (-4*q^2+11*q+2)/(q^2-2*q+1)
]
*/
// From the above, we consider the next three cases:
//// CaseB-2-1:q-4=0,
//// CaseB-2-2:pp=0.
//// CaseB-2-1 : q-4=0 :
 *
If q=4, by Maple we have
kai:= [-2, -X12, X12, X12, X12, -1]
*/

//// CaseB-2-2:pp=0:
aa:=(2*q+1)*(q-1)^2;
bb:=-(q-1)*(2*q+1)*(q-4);
cc:=-13*q^3+30*q^2+6*q+4;
pp eq aa*X12^2+bb*X12+cc;
Disc:=bb^2-4*aa*cc;
Disc eq 27*q^2*(2*q+1)*(2*q-5)*(q-1)^2;
```

```
// Since Disc ge 0, pp always has two solutions.

/*
pp1:=(1/2)*(2*q^2-7*q-4+3*sqrt(12*q^4-24*q^3
     -15*q^2))/((2*q+1)*(q-1));
pp2:=-(1/2)*(-2*q^2+7*q+4+3*sqrt(12*q^4-24*q^3
     -15*q^2))/((2*q+1)*(q-1))
for q from 3 to 20 do
 print([q, evalf(pp1), evalf(pp2)]);
end do;
         [3, 1.222970758, -1.722970758]
         [4, 2.000000000, -2.000000000]
         [5, 2.314528280, -2.064528280]
         [6, 2.487760074, -2.087760074]
         [7, 2.597871377, -2.097871377]
         [8, 2.674161873, -2.102733301]
         [9, 2.730182598, -2.105182598]
        [10, 2.773083515, -2.106416849]
        [11, 2.806999761, -2.106999761]
        [12, 2.834491270, -2.107218542]
        [13, 2.857228502, -2.107228502]
        [14, 2.876348298, -2.107117528]
        [15, 2.892651407, -2.106937121]
        [16, 2.906718246, -2.106718246]
        [17, 2.918979867, -2.106479867]
        [18, 2.929763136, -2.106233724]
        [19, 2.939320439, -2.105987105]
        [20, 2.947849761, -2.105744497]
```

```
*/
```

□

3.2.3　Case B-3: $(q+2)X_{12}+(2q+1)X_{13}+3q-6=0$.

Let $X_{13}=-\dfrac{(q+2)X_{12}+3q-6}{2q+1}$. Then we have the following:

Lemma 17.

Let $X_{23}=-\dfrac{(q+2)(X_{12}+X_{13})+3q-6}{q-1}$, $X_{13}=-\dfrac{(q+2)X_{12}+3q-6}{2q+1}$. *Then we have the following:*

$$(X_{12}-2)(X_{12}+2q-4)((q+2)X_{12}-2q^2+6q-4)=0. \qquad (34)$$

Proof.
```
//// CaseB-3:(q+2)*X12+(2*q+1)*X13+3*q-6=0:
P1<X12>:=PolynomialRing(F);
X13:=-(X12*q+2*X12-6+3*q)/(2*q+1);
X23:=-(X12*q+2*X12-6+2*X13+3*q+X13*q)/(q-1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
      -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
        +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
        +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
        +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
        +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
        +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4, ]*X12
        +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
```

```
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
        +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
        +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
        +2*(X2*X3+X12*X13);
g12-g24 eq (q+2)*(X12-2)*(X12+2*q-4)*((q+2)*X12
        -2*q^2+6*q-4)/(2*q+1)^2;
```

□

From (34) we consider the next three cases:

$$\begin{cases} \text{Case B-3-1:} \quad X_{12}-2=0, \\ \text{Case B-3-2:} \quad X_{12}+2q-4=0, \\ \text{Case B-3-3:} \quad (q+2)X_{12}-2q^2+6q-4=0. \end{cases}$$

For these, we have the following:

Lemma 18.

Let $X_{23}=-\dfrac{(q+2)(X_{12}+X_{13})+3q-6}{q-1}$, $X_{13}=-\dfrac{(q+2)X_{12}+3q-6}{2q+1}$. *Then Case B-3-2 does not satisfy a type II matrix. Cases B-3-1, B-3-3 are reduced to the following, respectively:*

(1) $X_1=X_2=-q+2$, $X_3=\dfrac{q^2+2}{2q+1}$, $X_{12}=2$, $X_{13}=X_{23}=\dfrac{-5q+1}{2q+1}$,

(2) $X_1=X_2=X_{13}=X_{23}=-2,\ X_3=X_{12}=2.$

Proof.
```
//// CaseB-3-1:X12－2＝0:
CaseB31:＝hom＜P1－＞F|[2]＞;
eq7:＝[g12,g13,g23,g24,h12,h13,h23];
&and[e @ Case B31 eq 0:e in eq7];
kai:＝[X1,X2,X3,X12,X13,X23];
kaiB31:＝[e @ Case B31:e in kai];
print “kaiB31”;
kaiB31;
/*
[
    －q＋2,
    －q＋2,
    (1/2*q^2＋1)/(q＋1/2),
    2,
    (－5/2*q＋1)/(q＋1/2),
    (－5/2*q＋1)/(q＋1/2)
]
*/
// X1＝X2＝－q＋2, X3＝(1/2*q^2＋1)/(q＋1/2), X12＝2, X13＝
    X23＝(－5/2*q＋1)/(q＋1/2).

//// CaseB-3-2:X12＋2*q－4＝0:
CaseB32:＝hom＜P1－＞F|[－2*q＋4]＞;
g12 @ CaseB32 eq (2*(q－1))*(q^2－2*q－2);
// This is a contradiction.
```

```
//// CaseB-3-3:(q+2)*X12-2*q^2+6*q-4=0:
CaseB33:=hom<P1->F|[(2*q^2-6*q+4)/(q+2)]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
caseB33:=[e @ Case B33:e in eq7];case B33;
/*
[
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (2*q^8-24*q^7+100*q^6-152*q^5+128*q^3)/
      (q^3+6*q^2+12*q+8),
    (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4),
    (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4)
]
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4)
    =-2*q^3*(q-4)^2/(q+2)^2;
*/
kai:=[X1,X2,X3,X12,X13,X23];
kaiB33:=[e @ Case B33:e in kai];
print "kaiB33;
kaiB33;
/*
[
    -q+2,
    -q+2,
    (q^3-4*q^2+2*q+4)/(q+2),
    (2*q^2-6*q+4)/(q+2),
```

```
    −q+2,
    −q+2
]
[−2, −2, 2, 2, −2, −2];
X1=X2=X13=X23=−2, X3=X12=2.
*/
```

□

3.3 Case C: $(q-1)^2(X_{12}+X_{13}+X_{23})+3(q^2-2q-2)=0$.

In this subsection, we assume that

$X_{23}=-\dfrac{(q-1)^2(X_{12}+X_{13})+3(q^2-2q-2)}{(q-1)^2}$. Then, we have the following:

Lemma 19. *Let* $X_{23}=-\dfrac{(q-1)^2(X_{12}+X_{13})+3(q^2-2q-2)}{(q-1)^2}$. *We set*

$$a=(q-1)^4(X_{12}+2),$$
$$b=(q-1)^2(X_{12}+2)(q^2X_{12}-2qX_{12}+X_{12}-6q+3q^2-6),$$
$$c=2(q-1)^4X_{12}^2+(6(q^2-2q-2))(q-1)^2X_{12}$$
$$+(q+2)(q-4)(5q^2-10q-4).$$

Then we set

$$p=aX_{13}^2+bX_{13}+c.$$

If $p=0$, *then there infinite type II matrices for* $q\geq 3$.

Proof.
```
// CaseC:(q−1)^2*(X12+X13+X23)+3*(q^2−2*q
    −2)=0:
P2<X12,X13>:=PolynomialRing(F,2);
X23:=−(X12*q^2−2*X12*q+X12−6*q+3*q^2+X13*q^2
      +X13−6−2*X13*q)/(q^2−2*q+1);
```

```
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
    -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
 e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
     +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
     +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
 e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
     +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
     +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
 e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
     +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
     +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
     +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
     +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
     +2*(X2*X3+X12*X13);
// Then we have the following:
pp:=32+18*X13*q^2+12*X13*q+18*X12*q^2+12*X12*q
    +2*X12^2+2*X13^2-4*X12*X13-24*q^2+88*q
    -12*X12-12*X13-20*q^3+5*q^4-24*X13*q^3
    -24*X12*q^3+6*q^4*X13+6*q^4*X12+2*X13^2*q^4
```

```
    －8*X13^2*q^3＋12*X13^2*q^2－8*X13^2*q
    ＋2*X12^2*q^4－8*X12^2*q^3＋12*X12^2*q^2
    －8*X12^2*q＋X12*X13^2＋X12^2*X13＋21*X13*q^2*X12
    －2*X13*q*X12－20*X13*q^3*X12＋5*X13*q^4*X12
    ＋X12*X13^2*q^4－4*X12*X13^2*q^3＋6*X12*X13^2*q^2
    －4*X12*X13^2*q＋6*X13*q^2*X12^2－4*X13*q*X12^2
    －4*X13*q^3*X12^2＋X13*q^4*X12^2;

aa:＝(q－1)^4*(2＋X12);
bb:＝(q－1)^2*(2＋X12)*(X12*q^2－2*X12*q＋X12－6*q
    ＋3*q^2－6);
cc:＝2*(q－1)^4*X12^2＋(6*(q^2－2*q－2))*(q－1)^2*X12
    ＋(q＋2)*(q－4)*(5*q^2－10*q－4);
pp eq aa*X13^2＋bb*X13＋cc;

g12 eq g13;
g12 eq g23;
h12 eq h13;
h12 eq h23;

g12－g24 eq (1/9)*(q＋2)*(q－4)*pp/(q－1)^4;
g12 eq 1/9*pp/(q－1)^2;
h12 eq－2/9*pp/(q－1)^2;

kai:＝[X1,X2,X3,X12,X13,X23];
kai;
/*
[
```

```
    (−1/3*q+1/3)*X12+(−1/3*q+1/3)*X13−2/3*q+2/3,
    (1/3*q−1/3)*X13+(1/3*q^2−2/3*q−8/3)/(q−1),
    (1/3*q−1/3)*X12+(1/3*q^2−2/3*q−8/3)/(q−1),
    X12,
    X13,
    −X12−X13+(−3*q^2+6*q+6)/(q^2−2*q+1)
]
*/
Disc:=bb^2−4*aa*cc;
Disc eq (q−1)^4*(X12−2)*(2+X12)*((q−1)^2*X12+q^2
    +4*q−14)
*((q−1)^2*X12+q^2−8*q−2);
// Since−2<X12<2, (X12−2)*(2+X12)<0.
// We set f1:=(q−1)^2*X12+q^2+4*q−14;
          f2:=(q−1)^2*X12+q^2−8*q−2;
// To show Disc ge 0, it is only necessary to require that
when f1*f2 ge?
// q=2, f1=−2, f2=−14,
// q=3, f1=7, f2=−17,
// q=4, f1=18, f2=−18.
// The answer is for q ge 3.
```

□

4 Appendix

```
// Classification of complex Hadamard matrices on
// amorphous pseudocyclic association schemes of three
classes.
// We always assume that q ne \pm 1. Because, if q=\pm 1,
```

```
then n=1.

F<q>:=FunctionField(Rationals());

r:=(q-1)/3;
s:=r;
t:=-(1+r+s);
k:=-(r*s+s*t+t*r);
n:=3*k+1;
n eq q^2;

P:=Matrix(F,4,4,[
  1,k,k,k,
  1,t,s,r,
  1,r,t,s,
  1, s, r, t
]);
print P;
/*
[        1  1/3*q^2-1/3  1/3*q^2-1/3  1/3*q^2-1/3]
[        1  -2/3*q-1/3    1/3*q-1/3     1/3*q-1/3]
[        1    1/3*q-1/3  -2/3*q-1/3     1/3*q-1/3]
[        1    1/3*q-1/3    1/3*q-1/3   -2/3*q-1/3]
*/
print n;

P6<X1,X2,X3,X12,X13,X23>:=PolynomialRing(F,6);
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
```

```
    +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
    +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
 e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
    +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
    +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
 e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
    +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
    +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
    +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
    +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
    +2*(X2*X3+X12*X13);
eq10:=[e1,e2,e3,g12,g13,g23,g24,h12,h13,h23];
eq10;
/*
[
  (-2/3*q-1/3)*X1+(1/3*q-1/3)*X2+(1/3*q-1/3)*X3
    +(-2/9*q^2+1/9*q+1/9)*X12+(-2/9*q^2+1/9*q
    +1/9)*X13+(1/9*q^2-2/9*q+1/9)*X23-1/3*q^2
    +4/3,
  (1/3*q-1/3)*X1+(-2/3*q-1/3)*X2+(1/3*q-1/3)*X3
    +(-2/9*q^2+1/9*q+1/9)*X12+(1/9*q^2-2/9*q
```

```
        +1/9)*X13+(-2/9*q^2+1/9*q+1/9)*X23-1/3*q^2
        +4/3,
    (1/3*q-1/3)*X1+(1/3*q-1/3)*X2+(-2/3*q-1/3)*X3
        +(1/9*q^2-2/9*q+1/9)*X12+(-2/9*q^2+1/9*q
        +1/9)*X13+(-2/9*q^2+1/9*q+1/9)*X23
        -1/3*q^2+4/3,
    X1^2-X1*X2*X12+X2^2+X12^2-4,
    X1^2-X1*X3*X13+X3^2+X13^2-4,
    X2^2-X2*X3*X23+X3^2+X23^2-4,
    X12^2-X12*X13*X23+X13^2+X23^2-4,
    2*X1*X2-X1*X3*X23-X2*X3*X13+X3^2*X12-4*X12
        +2*X13*X23,
    -X1*X2*X23+2*X1*X3+X2^2*X13-X2*X3*X12
        +2*X12*X23-4*X13,
    X1^2*X23-X1*X2*X13-X1*X3*X12+2*X2*X3+2*X12*X13
        -4*X23
]
*/
e1-e2 eq-(1/3)*q*(X13*q-X23*q+3*X1-3*X2-X13+X23);
e1-e3 eq-(1/3)*q*(X12*q-X23*q+3*X1-3*X3-X12+X23);
// From e1-e2, we have X2:=X1+(1/3)*X13*q-(1/3)*X13
    -(1/3)*X23*q+(1/3)*X23;
// From e1-e3, we have X3:=X1+(1/3)*X12*q-(1/3)*X12
    -(1/3)*X23*q+(1/3)*X23;

// In what follows, we set
// X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q
    +(1/3)*X23;
```

```
// X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q
    +(1/3)*X23;

P4<X1,X12,X13,X23>:=PolynomialRing(F,4);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
h1:=hom<P6->P4|[X1,X2,X3,X12,X13,X23]>;
e11:=e1 @ h1;
e21:=e2 @ h1;
e31:=e3 @ h1;
e11 eq e21;
e11 eq e31;
e11 eq-(1/9)*X13*q^2-(1/9)*X13*q-(1/9)*X23*q^2
    +(2/9)*X23*q-(1/9)*X12*q^2-(1/9)*X12*q-(1/3)*q^2
    +4/3-X1+(2/9)*X12+(2/9)*X13-(1/9)*X23;
// From this eqaution, we have
// X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
    -1/3*(q-2)*(q+2);
// In this stage, we have e1=e2=e3=0.

// Moreover, we set
// X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
    -1/3*(q-2)*(q+2);;

P3<X12,X13,X23>:=PolynomialRing(F,3);
// From our assumption (q ne \pm 1), the next equations X12,
    X13,X23  always hold.
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
```

```
    -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
      +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
      +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
      +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
      +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
      +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
      +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
     +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
     +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
     +2*(X2*X3+X12*X13);

g12-g13 eq-(1/81*(q+2))*(X12-X13)*((q+2)*X12
    +(q+2)*X13+(q-1)*X23+3*q-6)
*((q-1)^2*(X12+X13+X23)+3*(q^2-2*q-2));
// From this, we consider the next three cases:
// CaseA:X12-X13=0,
```

```
// CaseB:(q+2)*X12+(q+2)*X13+(q-1)*X23+3*q-6=0,
// CaseC:(q-1)^2*(X12+X13+X23)+3*(q^2-2*q-2)=0.

// CaseA:X12-X13=0:
P2<X12, X23>:=PolynomialRing(F,2);
X13:=X12;
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
     -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
       +P[2,3]*X2+P[2,4]*X3+ P[2,2]*P[2,3]*X12
       +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
       +P[3,3]*X2+P[3,4]*X3+ P[3,2]*P[3,3]*X12
       +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
       +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
       +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
```

```
      +2*(X2*X3+X12*X13);
// Then we have the following:
g12-g23 eq-(1/81*(q+2))*(X12-X23)*((2*q+1)*X12
      +(q+2)*X23+3*q-6)
*(2*(q-1)^2*X12+(q-1)^2*X23+3*q^2-6*q-6);
// From this, we consider the next three cases:

//// CaseA-1:X12-X23=0,
//// CaseA-2:(2*q+1)*X12+(q+2)*X23+3*q-6=0,
//// CaseA-3:2*(q-1)^2*X12+(q-1)^2*X23+3*q^2-6*q
                -6=0.

//// CaseA-1:X12-X23=0:
P1<X12>:=PolynomialRing(F);
X13:=X12;
X23:=X12;
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
      -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
      +P[2,3]*X2+P[2,4]*X3+ P[2,2]*P[2,3]*X12
      +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
      +P[3,3]*X2+P[3,4]*X3+ P[3,2]*P[3,3]*X12
      +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
      +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
```

```
    +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
    +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
    +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
    +2*(X2*X3+X12*X13);
g12-g24 eq-(1/9*(q-2))*(q+2)*(X12+1)*(X12-2)*((q^2
    +2)*X12-4+q^2);
// From this, we consider the next cases:
//// CaseA-1-1:X12+1=0,
//// CaseA-1-2:X12-2=0,
//// CaseA-1-3:(q^2+2)*X12-4+q^2=0.

//// CaseA-1-1:X12+1=0:
CaseA11:=hom<P1->F|[-1]>;
h12 @ Case A11 eq 9;
// This is a contradiction.
//// CaseA-1-2:X12-2=0:
CaseA12:=hom<P1->F|[2]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
&and[e @ CaseA12 eq 0:e in eq7];
X1 @ CaseA12 eq-q^2+2;
X2 @ CaseA12 eq-q^2+2;
```

```
X3 @ CaseA12 eq－q^2＋2;
X12 @ CaseA12 eq 2;
X13 @ CaseA12 eq 2;
X23 @ CaseA12 eq 2;
// X1＝X2＝X3＝－q^2＋2,X12＝X13＝X23＝2.
//// CaseA-1-3:(q^2＋2)*X12－4＋q^2＝0:
CaseA13:＝hom<P1－>F|[－(－4＋q^2)/(q^2＋2)]>;
g12 @ CaseA13 eq－54*q^4/(q^6＋6*q^4＋12*q^2＋8);
// This is a contradiction.

//// CaseA-2:(2*q＋1)*X12＋(q＋2)*X23＋3*q－6＝0:
P1<X12>:＝PolynomialRing(F);
X13:＝X12;
X23:＝－(3*q＋2*X12*q－6＋X12)/(q＋2);
X1:＝－1/9*(q＋2)*(q－1)*(X12＋X13)－1/9*(q－1)^2*X23
    －1/3*(q－2)*(q＋2);
X2:＝X1＋(1/3)*X13*q－(1/3)*X13－(1/3)*X23*q＋(1/3)*X23;
X3:＝X1＋(1/3)*X12*q－(1/3)*X12－(1/3)*X23*q＋(1/3)*X23;
  e1:＝1＋P[2,2]^2＋P[2,3]^2＋P[2,4]^2＋P[2,2]*X1
      ＋P[2,3]*X2＋P[2,4]*X3＋ P[2,2]*P[2,3]*X12
      ＋P[2,2]*P[2,4]*X13＋P[2,3]*P[2,4]*X23－n;
  e2:＝1＋P[3,2]^2＋P[3,3]^2＋P[3,4]^2＋P[3,2]*X1
      ＋P[3,3]*X2＋P[3,4]*X3＋ P[3,2]*P[3,3]*X12
      ＋P[3,2]*P[3,4]*X13＋P[3,3]*P[3,4]*X23－n;
  e3:＝1＋P[4,2]^2＋P[4,3]^2＋P[4,4]^2＋P[4,2]*X1
      ＋P[4,3]*X2＋P[4,4]*X3＋P[4,2]*P[4,3]*X12
      ＋P[4,2]*P[4,4]*X13＋P[4,3]*P[4,4]*X23－n;
g12:＝X12^2－X1*X2*X12＋X1^2＋X2^2－4;
```

```
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
g12-g24 eq-((2*q+1)*X12+5*q-2)*(X12+2-q)*(X12-2
      +q)/(q+2);
//// CaseA-2-1:(2*q+1)*X12+5*q-2=0,
//// CaseA-2-2:X12+2-q=0,
//// CaseA-2-3:X12-2+q=0.

//// CaseA-2-1:(2*q+1)*X12+5*q-2=0:
CaseA21:=hom<P1->F|[-(5*q-2)/(1+2*q)]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
&and[e @ CaseA21 eq 0:e in eq7];
X1 @ CaseA21 eq (1/2*q^2+1)/(q+1/2);
X2 @ CaseA21 eq-q+2;
X3 @ CaseA21 eq-q+2;
X12 @ CaseA21 eq (-5/2*q+1)/(q+1/2);
X13 @ CaseA21 eq (-5/2*q+1)/(q+1/2);
X23 @ CaseA21 eq 2;
// X1=(1/2*q^2+1)/(q+1/2),X2=X3=-q+2,X12=X13
     =(-5/2*q+1)/(q+1/2),X23=2;
//// CaseA-2-2:X12+2-q=0:
```

```
Case A22:=hom<P1->F|[-2+q]>;
g12 @ CaseA22 eq (2*(q-1))*(q^2-2*q-2);
g13 @ CaseA22 eq (2*(q-1))*(q^2-2*q-2);
g23 @ CaseA22 eq (2*(q-1))*(q^2-2*q-2);
g24 @ CaseA22 eq (2*(q-1))*(q^2-2*q-2);
h12 @ CaseA22 eq (2*(q-1))*(-2+q)*(q^2-2*q-2);
h13 @ CaseA22 eq (2*(q-1))*(-2+q)*(q^2-2*q-2);
h23 @ CaseA22 eq-2*q*(q-1)*(-2+q)*(q^2-2*q-2);
// Discriminat of q^2-2*q-2 is minus.
//// CaseA-2-3:X12-2+q=0:
CaseA23:=hom<P1->F|[-q+2]>;
g12 @ CaseA23 eq-2*q^3*(q-4)^2/(q+2)^2;
// From this we have q=4.
CaseA232:=hom<P1->F|[-2]>;
g12 @ CaseA232;
g13 @ CaseA232;
g23 @ CaseA232;
g24 @ CaseA232;
h12 @ CaseA232;
h13 @ CaseA232;
h23 @ CaseA232;
caseA23:=[e @ CaseA232:e in eq7];caseA23;
/*
[
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^2-8*q+16)/(q^2+4*q+4),
```

```
    (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
    (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
    (2*q^5-8*q^4-27*q^3+88*q^2+80*q)/
      (q^3+6*q^2+12*q+8)
]
*/
// The next is due to Maple:
/*
A:=
[
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
    (q^2-8*q+16)/(q^2+4*q+4),
    (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
    (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
    (2*q^5-8*q^4-27*q^3+88*q^2+80*q)/
      (q^3+6*q^2+12*q+8)
];
q:=4;A;
*/
kai:=[X1,X2,X3,X12,X13,X23];
kaiA23:=[e @ CaseA232:e in kai];kaiA23;
/*
[
    3*q/(q+2),
    -q+2,
    -q+2,
```

```
   −2,
   −2,
   (q＋8)/(q＋2)
]
*/
//X1:＝2,X2:＝−2,X3:＝−2,X12:＝−2,X13:＝−2,X23:＝2.

////CaseA-3:2*(q−1)^2*X12＋(q−1)^2*X23＋3*q^2−6*q
     −6＝0:
P1<X12>:＝PolynomialRing(F);
X13:＝X12;
X23:＝−(2*X12*q^2−4*X12*q＋2*X12＋3*q^2−6*q−6)/
      (q^2−2*q＋1);
X1:＝−1/9*(q＋2)*(q−1)*(X12＋X13)−1/9*(q−1)^2*X23
     −1/3*(q−2)*(q＋2);
X2:＝X1＋(1/3)*X13*q−(1/3)*X13−(1/3)*X23*q＋(1/3)*X23;
X3:＝X1＋(1/3)*X12*q−(1/3)*X12−(1/3)*X23*q＋(1/3)*X23;
  e1:＝1＋P[2,2]^2＋P[2,3]^2＋P[2,4]^2＋P[2,2]*X1
       ＋P[2,3]*X2＋P[2,4]*X3＋P[2,2]*P[2,3]*X12
       ＋P[2,2]*P[2,4]*X13＋P[2,3]*P[2,4]*X23−n;
  e2:＝1＋P[3,2]^2＋P[3,3]^2＋P[3,4]^2＋P[3,2]*X1
       ＋P[3,3]*X2＋P[3,4]*X3＋P[3,2]*P[3,3]*X12
       ＋P[3,2]*P[3,4]*X13＋P[3,3]*P[3,4]*X23−n;
  e3:＝1＋P[4,2]^2＋P[4,3]^2＋P[4,4]^2＋P[4,2]*X1
       ＋P[4,3]*X2＋P[4,4]*X3＋P[4,2]*P[4,3]*X12
       ＋P[4,2]*P[4,4]*X13＋P[4,3]*P[4,4]*X23−n;
g12:＝X12^2−X1*X2*X12＋X1^2＋X2^2−4;
g13:＝X13^2−X1*X3*X13＋X1^2＋X3^2−4;
```

```
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
g12-g24 eq (1/9)*(q+2)*(q-4)*((q-1)*X12-4+q)*(2*(q
      -1)^2*X12+5*q^2-10*q-4)
*((q-1)*X12+2+q)/(q-1)^4;
// From this, we consider the next three cases:
////CaseA-3-1:q-4=0,
////CaseA-3-2:(q-1)*X12-4+q=0,
////CaseA-3-3:2*(q-1)^2*X12+5*q^2-10*q-4=0,
////CaseA-3-4:(q-1)*X12+2+q=0.

////CaseA-3-1:q-4=0:
CaseA31:=hom<P1->F|[4]>;
g12 @ CaseA31 eq (1/9)*(5*q-8)*(5*q-2)*(13*q^2-26*q
     +4)/(q-1)^2;
// Since q is a positive, this is a contradiction.
////CaseA-3-2:(q-1)*X12-4+q=0:
CaseA32:=hom<P1->F|[-(q-4)/(q-1)]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
&and[e @ CaseA32 eq 0:e in eq7];
kai:=[X1, X2, X3, X12, X13, X23];
kaiA32:=[e @ CaseA32:e in kai];
```

```
print “kaiA32”;
kaiA32;
/*
[
    －2,
    (q－4)/(q－1),
    (q－4)/(q－1),
    (－q＋4)/(q－1),
    (－q＋4)/(q－1),
    (－q^2－4*q＋14)/(q^2－2*q＋1)
]
*/
// X1＝－2, X2＝X3＝(q－4)/(q－1),X12＝X13＝(－q＋4)/(q－1),
      X23＝(－q^2－4*q＋14)/(q^2－2*q＋1).
////CaseA-3-3:2*X12*q^2－4*X12*q＋2*X12＋5*q^2－10*q
      －4＝0:
CaseA33:＝hom<P1－>F|[－(1/2)*(5*q^2－10*q－4)/
      (q^2－2*q＋1)]>;
eq7:＝[g12,g13,g23,g24,h12,h13,h23];
&and[e @ CaseA33 eq 0:e in eq7];
kai:＝[X1,X2,X3,X12,X13,X23];
kaiA33:＝[e @ CaseA33:e in kai];
print “kaiA33”;
kaiA33;
/*
[
    (q^2－2*q－2)/(q－1),
    (－1/2*q^2＋q－2)/(q－1),
```

```
    (−1/2*q^2+q−2)/(q−1),
    (−5/2*q^2+5*q+2)/(q^2−2*q+1),
    (−5/2*q^2+5*q+2)/(q^2−2*q+1),
    2
]
*/
// X1=(q^2−2*q−2)/(q−1),X2=X3=(−1/2*q^2+q−2)/
      (q−1),X12=X13=(−5/2*q^2+5*q+2)/
      (q^2−2*q+1),X23=2.
////CaseA-3-4:X12*q−X12+2+q=0:
CaseA34:=hom<P1−>F|[−(q+2)/(q−1)]>;
&and[e @ CaseA34 eq 0:e in eq7];
kai:=[X1,X2,X3,X12,X13,X23];
kaiA34:=[e @ CaseA34:e in kai];
print "kaiA34";
kaiA34;
/*
[
    2,
    (−q−2)/(q−1),
    (−q−2)/(q−1),
    (−q−2)/(q−1),
    (−q−2)/(q−1),
    (−q^2+8*q+2)/(q^2−2*q+1)
]
*/
// X1=2, X2=X3=X12=X13=(−q−2)/(q−1),
      X23=(−q^2+8*q+2)/(q^2−2*q+1).
```

```
// CaseB:(q+2)*X12+(q+2)*X13+(q-1)*X23+3*q-6=0:
P2<X12,X13>:=PolynomialRing(F,2);
X23:=-(X12*q+2*X12-6+2*X13+3*q+X13*q)/(q-1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
      -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
       +P[2,3]*X2+P[2,4]*X3+ P[2,2]*P[2,3]*X12
       +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
       +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
       +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
       +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
       +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
// Then we have the following:
g12-g23 eq-(1/9)*(q+2)*((2*q+1)*X12+(q+2)*X13
```

```
      +3*q-6)*
((q-1)*X12+(q-1)*X13-q+4)*((q+2)*X12+(2*q+1)*X13
      +3*q-6)/(q-1)^2;
// From this we consider the next three cases:
//// CaseB-1:(2*q+1)*X12+(q+2)*X13+3*q-6=0,
//// CaseB-2:(q-1)*X12+(q-1)*X13-q+4=0,
//// CaseB-3:(q+2)*X12+(2*q+1)*X13+3*q-6=0.

//// CaseB-1:(2*q+1)*X12+(q+2)*X13+3*q-6=0:
P1<X12>:=PolynomialRing(F);
X13:=-(2*X12*q+X12-6+3*q)/(q+2);
X23:=-(X12*q+2*X12-6+2*X13+3*q+X13*q)/(q-1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
     -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
      +P[2,3]*X2+P[2,4]*X3+ P[2,2]*P[2,3]*X12
      +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
      +P[3,3]*X2+P[3,4]*X3+ P[3,2]*P[3,3]*X12
      +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
      +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
      +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
```

```
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
       +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
       +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
       +2*(X2*X3+X12*X13);
g12-g24 eq-(X12-2+q)*(X12+2-q)*((2*q+1)*X12-2
       +5*q)/(q+2);
// From this we consider the next three cases:
//// CaseB-1-1:X12-2+q=0,
//// CaseB-1-2:X12+2-q=0,
//// CaseB-1-3:(2*q+1)*X12-2+5*q=0.

//// CaseB-1-1:X12-2+q=0:
CaseB11:=hom<P1->F|[-q+2]>;
g12 @ CaseB11 eq-2*(q-4)^2*q^3/(q+2)^2;
// From this we have q=4.
eq7:=[g12,g13,g23,g24,h12,h13,h23];
caseB11:=[e @ CaseB11:e in eq7];caseB11;
/*
[
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4),
    (2*q^8-24*q^7+100*q^6-152*q^5+128*q^3)/
```

```
      (q^3+6*q^2+12*q+8),
    (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4)
]
*/
/*
By Maple,
A:=
[
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4),
    (2*q^8-24*q^7+100*q^6-152*q^5+128*q^3)/
      (q^3+6*q^2+12*q+8),
    (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4)
];
q:=4; A;
[0, 0, 0, 0, 0, 0, 0];
*/
kai:=[X1,X2,X3,X12,X13,X23];
kaiB11:=[e @ CaseB11:e in kai];
print "kaiB11";
kaiB11;
/*
[
    -q+2,
    (q^3-4*q^2+2*q+4)/(q+2),
```

```
    －q＋2,
    －q＋2,
    (2*q^2－6*q＋4)/(q＋2),
    －q＋2
]
[－2, 2,－2,－2, 2,－2];
X1＝X3＝X12＝X23＝－2, X2＝X13＝2.
*/

//// CaseB-1-2:X12＋2－q＝0:
CaseB12:＝hom<P1－>F|[q－2]>;
g12 @ CaseB12 eq (2*(q－1))*(q^2－2*q－2);
// From this we have q＝1.
//// CaseB-1-3:2*X12*q＋X12－2＋5*q＝0:
CaseB13:＝hom<P1－>F|[－(－2＋5*q)/(2*q＋1)]>;
&and[e @ CaseB13 eq 0:e in eq7];
kai:＝[X1,X2,X3,X12,X13,X23];
kaiB13:＝[e @ CaseB13:e in kai];
print "kaiB13";
kaiB13;
/*
[
    －q＋2,
    (1/2*q^2＋1)/(q＋1/2),
    －q＋2,
    (－5/2*q＋1)/(q＋1/2),
    2,
    (－5/2*q＋1)/(q＋1/2)
```

```
]
X1＝X3＝－q＋2, X2＝(1/2*q^2＋1)/(q＋1/2), X12＝X23
    ＝(－5/2*q＋1)/(q＋1/2), X13＝2.
*/


//// CaseB-2:(q－1)*X12＋(q－1)*X13－q＋4＝0,
P1<X12>:＝PolynomialRing(F);
X13:＝－(X12*q－X12＋4－q)/(q－1);
X23:＝－(X12*q＋2*X12－6＋2*X13＋3*q＋X13*q)/(q－1);
X1:＝－1/9*(q＋2)*(q－1)*(X12＋X13)－1/9*(q－1)^2*X23
      －1/3*(q－2)*(q＋2);
X2:＝X1＋(1/3)*X13*q－(1/3)*X13－(1/3)*X23*q＋(1/3)*X23;
X3:＝X1＋(1/3)*X12*q－(1/3)*X12－(1/3)*X23*q＋(1/3)*X23;
  e1:＝1＋P[2,2]^2＋P[2,3]^2＋P[2,4]^2＋P[2,2]*X1
       ＋P[2,3]*X2＋P[2,4]*X3＋P[2,2]*P[2,3]*X12
       ＋P[2,2]*P[2,4]*X13＋P[2,3]*P[2,4]*X23－n;
  e2:＝1＋P[3,2]^2＋P[3,3]^2＋P[3,4]^2＋P[3,2]*X1
       ＋P[3,3]*X2＋P[3,4]*X3＋P[3,2]*P[3,3]*X12
       ＋P[3,2]*P[3,4]*X13＋P[3,3]*P[3,4]*X23－n;
  e3:＝1＋P[4,2]^2＋P[4,3]^2＋P[4,4]^2＋P[4,2]*X1
       ＋P[4,3]*X2＋P[4,4]*X3＋P[4,2]*P[4,3]*X12
       ＋P[4,2]*P[4,4]*X13＋P[4,3]*P[4,4]*X23－n;
g12:＝X12^2－X1*X2*X12＋X1^2＋X2^2－4;
g13:＝X13^2－X1*X3*X13＋X1^2＋X3^2－4;
g23:＝X23^2－X2*X3*X23＋X2^2＋X3^2－4;
g24:＝X12^2＋X13^2＋X23^2－X12*X13*X23－4;
h12:＝(X3^2－4)*X12－X3*(X1*X23＋X2*X13)
       ＋2*(X1*X2＋X13*X23);
```

```
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
g12-g24 eq-(1/9)*(q+2)*(q-4)^2*((2*q+1)*(q
      -1)^2*X12^2-(q-1)*(2*q+1)*(q-4)*X12
      -13*q^3+30*q^2+6*q+4)/(q-1)^4;
// From this we consider the next three cases:
//// CaseB-2-1:q-4=0,
//// CaseB-2-2:(2*q+1)*(q-1)^2*X12^2
       -(q-1)*(2*q+1)*(q-4)*X12-13*q^3+30*q^2
       +6*q+4=0.

print "general:";
//// In general:
g12 eq g13;
g12 eq g23;
h12 eq h13;
h12 eq h23;
pp:=(2*q+1)*(q-1)^2*X12^2-(q-1)*(2*q+1)*(q-4)*X12
     -13*q^3+30*q^2+6*q+4;
g12 eq-(1/9)*(q-4)*pp/(q-1)^2;
g24 eq-(q-4)*pp/(q-1)^4;
h12 eq (2/9)*(q-4)*pp/(q-1)^2;

kai:=[X1,X2,X3,X12,X13,X23];
kai;
/*
```

```
[
    -q+2,
    (-1/3*q+1/3)*X12+(2/3*q^2-7/3*q-4/3)/(q-1),
    (1/3*q-1/3)*X12+(1/3*q^2-2/3*q-8/3)/(q-1),
    X12,
    -X12+(q-4)/(q-1),
    (-4*q^2+11*q+2)/(q^2-2*q+1)
]
*/

// From the above, we consider the next three cases:
//// CaseB-2-1:q-4=0,
//// CaseB-2-2:pp=0.

//// CaseB-2-1:q-4=0:
/*
If q=4, by Maple we have
kai := [-2, -X12, X12, X12, -X12, -2];
*/

//// CaseB-2-2:pp=0:
aa:=(2*q+1)*(q-1)^2;
bb:=-(q-1)*(2*q+1)*(q-4);
cc:=-13*q^3+30*q^2+6*q+4;
pp eq aa*X12^2+bb*X12+cc;
Disc:=bb^2-4*aa*cc;
Disc eq 27*q^2*(2*q+1)*(2*q-5)*(q-1)^2;
// Since Disc ge 0, pp always has two solutions.
```

```
/*
pp1:=(1/2)*(2*q^2-7*q-4+3*sqrt(12*q^4-24*q^3
      -15*q^2))/((2*q+1)*(q-1));
pp2:=-(1/2)*(-2*q^2+7*q+4+3*sqrt(12*q^4-24*q^3
      -15*q^2))/((2*q+1)*(q-1))
for q from 3 to 20 do
  print([q,evalf(pp1),evalf(pp2)]);
end do;
            [3, 1.222970758, -1.722970758]
            [4, 2.000000000, -2.000000000]
            [5, 2.314528280, -2.064528280]
            [6, 2.487760074, -2.087760074]
            [7, 2.597871377, -2.097871377]
            [8, 2.674161873, -2.102733301]
            [9, 2.730182598, -2.105182598]
           [10, 2.773083515, -2.106416849]
           [11, 2.806999761, -2.106999761]
           [12, 2.834491270, -2.107218542]
           [13, 2.857228502, -2.107228502]
           [14, 2.876348298, -2.107117528]
           [15, 2.892651407, -2.106937121]
           [16, 2.906718246, -2.106718246]
           [17, 2.918979867, -2.106479867]
           [18, 2.929763136, -2.106233724]
           [19, 2.939320439, -2.105987105]
           [20, 2.947849761, -2.105744497]
*/
```

```
//// CaseB-3:(q+2)*X12+(2*q+1)*X13+3*q-6=0:
P1<X12>:=PolynomialRing(F);
X13:=-(X12*q+2*X12-6+3*q)/(2*q+1);
X23:=-(X12*q+2*X12-6+2*X13+3*q+X13*q)/(q-1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
     -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
       +P[2,3]*X2+P[2,4]*X3+ P[2,2]*P[2,3]*X12
       +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
       +P[3,3]*X2+P[3,4]*X3+ P[3,2]*P[3,3]*X12
       +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
       +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
       +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
      +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
      +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
      +2*(X2*X3+X12*X13);
g12-g24 eq (q+2)*(X12-2)*(X12+2*q-4)*((q+2)*X12
```

```
    −2*q^2＋6*q−4)/(2*q＋1)^2;
// From this we consider the next three cases:
//// CaseB-3-1:X12−2＝0,
//// CaseB-3-2:X12＋2*q−4＝0,
//// CaseB-3-3:(q＋2)*X12−2*q^2＋6*q−4＝0.

//// CaseB-3-1:X12−2＝0:
CaseB31:＝hom<P1−>F|[2]>;
eq7:＝[g12,g13,g23,g24,h12,h13,h23];
&and[e @ CaseB31 eq 0:e in eq7];
kai:＝[X1,X2,X3,X12,X13,X23];
kaiB31:＝[e @ CaseB31:e in kai];
print "kaiB31";
kaiB31;
/*
[
   −q＋2,
   −q＋2,
   (1/2*q^2＋1)/(q＋1/2),
   2,
   (−5/2*q＋1)/(q＋1/2),
   (−5/2*q＋1)/(q＋1/2)
]
*/
// X1＝X2＝−q＋2, X3＝(1/2*q^2＋1)/(q＋1/2), X12＝2,
     X13＝X23＝(−5/2*q＋1)/(q＋1/2).

//// CaseB-3-2:X12＋2*q−4＝0:
```

```
CaseB32:=hom<P1->F|[-2*q+4]>;
g12 @ CaseB32 eq (2*(q-1))*(q^2-2*q-2);
// This is a contradiction.

//// CaseB-3-3:(q+2)*X12-2*q^2+6*q-4=0:
CaseB33:=hom<P1->F|[(2*q^2-6*q+4)/(q+2)]>;
eq7:=[g12,g13,g23,g24,h12,h13,h23];
caseB33:=[e @ CaseB33:e in eq7];caseB33;
/*
[
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
    (2*q^8-24*q^7+100*q^6-152*q^5+128*q^3)/
      (q^3+6*q^2+12*q+8),
    (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4),
    (2*q^6-20*q^5+64*q^4-64*q^3)/(q^2+4*q+4)
]
(-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4)=
    -2*q^3*(q-4)^2/(q+2)^2;
*/
kai:=[X1,X2,X3,X12,X13,X23];
kaiB33:=[e @ CaseB33:e in kai];
print "kaiB33";
kaiB33;
/*
[
```

```
    -q+2,
    -q+2,
    (q^3-4*q^2+2*q+4)/(q+2),
    (2*q^2-6*q+4)/(q+2),
    -q+2,
    -q+2
]
[-2, -2, 2, 2, -2, -2];
X1=X2=X13=X23=-2, X3=X12=2.
*/

// CaseC:(q-1)^2*(X12+X13+X23)+3*(q^2-2*q-2)=0:
P2<X12, X13>:=PolynomialRing(F,2);
X23:=-(X12*q^2-2*X12*q+X12-6*q+3*q^2+X13*q^2
      +X13-6-2*X13*q)/(q^2-2*q+1);
X1:=-1/9*(q+2)*(q-1)*(X12+X13)-1/9*(q-1)^2*X23
     -1/3*(q-2)*(q+2);
X2:=X1+(1/3)*X13*q-(1/3)*X13-(1/3)*X23*q+(1/3)*X23;
X3:=X1+(1/3)*X12*q-(1/3)*X12-(1/3)*X23*q+(1/3)*X23;
  e1:=1+P[2,2]^2+P[2,3]^2+P[2,4]^2+P[2,2]*X1
       +P[2,3]*X2+P[2,4]*X3+P[2,2]*P[2,3]*X12
       +P[2,2]*P[2,4]*X13+P[2,3]*P[2,4]*X23-n;
  e2:=1+P[3,2]^2+P[3,3]^2+P[3,4]^2+P[3,2]*X1
       +P[3,3]*X2+P[3,4]*X3+P[3,2]*P[3,3]*X12
       +P[3,2]*P[3,4]*X13+P[3,3]*P[3,4]*X23-n;
  e3:=1+P[4,2]^2+P[4,3]^2+P[4,4]^2+P[4,2]*X1
       +P[4,3]*X2+P[4,4]*X3+P[4,2]*P[4,3]*X12
       +P[4,2]*P[4,4]*X13+P[4,3]*P[4,4]*X23-n;
```

```
g12:=X12^2-X1*X2*X12+X1^2+X2^2-4;
g13:=X13^2-X1*X3*X13+X1^2+X3^2-4;
g23:=X23^2-X2*X3*X23+X2^2+X3^2-4;
g24:=X12^2+X13^2+X23^2-X12*X13*X23-4;
h12:=(X3^2-4)*X12-X3*(X1*X23+X2*X13)
        +2*(X1*X2+X13*X23);
h13:=(X2^2-4)*X13-X2*(X1*X23+X3*X12)
        +2*(X1*X3+X12*X23);
h23:=(X1^2-4)*X23-X1*(X2*X13+X3*X12)
        +2*(X2*X3+X12*X13);
// Then we have the following:
pp:=32+18*X13*q^2+12*X13*q+18*X12*q^2+12*X12*q
      +2*X12^2+2*X13^2-4*X12*X13-24*q^2+88*q
      -12*X12-12*X13-20*q^3+5*q^4-24*X13*q^3
      -24*X12*q^3+6*q^4*X13+6*q^4*X12+2*X13^2*q^4
      -8*X13^2*q^3+12*X13^2*q^2-8*X13^2*q
      +2*X12^2*q^4-8*X12^2*q^3+12*X12^2*q^2
      -8*X12^2*q+X12*X13^2+X12^2*X13
      +21*X13*q^2*X12-2*X13*q*X12-20*X13*q^3*X12
      +5*X13*q^4*X12+X12*X13^2*q^4-4*X12*X13^2*q^3
      +6*X12*X13^2*q^2-4*X12*X13^2*q+6*X13*q^2*X12^2
      -4*X13*q*X12^2-4*X13*q^3*X12^2+X13*q^4*X12^2;

aa:=(q-1)^4*(2+X12);
bb:=(q-1)^2*(2+X12)*(X12*q^2-2*X12*q+X12-6*q
      +3*q^2-6);
cc:=2*(q-1)^4*X12^2+(6*(q^2-2*q-2))*(q-1)^2*X12
      +(q+2)*(q-4)*(5*q^2-10*q-4);
```

```
pp eq aa*X13^2＋bb*X13＋cc;

g12 eq g13;
g12 eq g23;
h12 eq h13;
h12 eq h23;

g12－g24 eq (1/9)*(q＋2)*(q－4)*pp/(q－1)^4;
g12 eq 1/9*pp/(q－1)^2;
h12 eq－2/9*pp/(q－1)^2;

kai:＝[X1,X2,X3,X12,X13,X23];
kai;
/*
[
    (－1/3*q＋1/3)*X12＋(－1/3*q＋1/3)*X13－2/3*q＋2/3,
    (1/3*q－1/3)*X13＋(1/3*q^2－2/3*q－8/3)/(q－1),
    (1/3*q－1/3)*X12＋(1/3*q^2－2/3*q－8/3)/(q－1),
    X12,
    X13,
    －X12－X13＋(－3*q^2＋6*q＋6)/(q^2－2*q＋1)
]
*/

Disc:＝bb^2－4*aa*cc;
Disc eq (q－1)^4*(X12－2)*(2＋X12)*((q－1)^2*X12＋q^2
    ＋4*q－14)*((q－1)^2*X12＋q^2－8*q－2);
// Since－2＜X12＜2, (X12－2)*(2＋X12)＜0.
```

```
// We set f1:=(q-1)^2*X12+q^2+4*q-14;
          f2:=(q-1)^2*X12+q^2-8*q-2;
// To show Disc ge 0, it is only necessary to require that
when f1*f2 ge?

// q=2, f1=-2, f2=-14,
// q=3, f1=7, f2=-17,
// q=4, f1=18, f2=-18.
// The answer is for q ge 3.

> load "magma/ComplexHadamard/Class3/takuya-original/
PseudocyclicScheme/amorphous/2013-02-07/new-matome.
magma.new";
Loading "magma/ComplexHadamard/Class3/takuya-original
/PseudocyclicScheme/amorphous/2013-02-07/new-matome.
magma.new"
true
[       1  1/3*q^2 -1/3  1/3*q^2 -1/3 1/3*q^2 -1/3]
[       1  -2/3*q -1/3    1/3*q -1/3    1/3*q -1/3]
[       1    1/3*q -1/3  -2/3*q -1/3    1/3*q -1/3]
[       1    1/3*q -1/3    1/3*q -1/3  -2/3*q -1/3]
q^2
[
   (-2/3*q-1/3)*X1+(1/3*q-1/3)*X2+(1/3*q-1/3)*X3
      +(-2/9*q^2+1/9*q+1/9)*X12 1/9*q+1/9)*X13
      +(1/9*q^2-2/9*q+1/9)*X23-1/3*q^2+4/3,
   (1/3*q-1/3)*X1+(-2/3*q-1/3)*X2+(1/3*q-1/3)*X3
      +(-2/9*q^2+1/9*q+1/9)*X12 2/9*q+1/9)*X13
```

```
    +(-2/9*q^2+1/9*q+1/9)*X23-1/3*q^2+4/3,
  (1/3*q-1/3)*X1+(1/3*q-1/3)*X2+(-2/3*q-1/3)*X3
    +(1/9*q^2-2/9*q+1/9)*X12
    1/9*q+1/9)*X13+(-2/9*q^2+1/9*q+1/9)*X23
    -1/3*q^2+4/3,
  X1^2-X1*X2*X12+X2^2+X12^2-4,
  X1^2-X1*X3*X13+X3^2+X13^2-4,
  X2^2-X2*X3*X23+X3^2+X23^2-4,
  X12^2-X12*X13*X23+X13^2+X23^2-4,
  2*X1*X2-X1*X3*X23-X2*X3*X13+X3^2*X12-4*X12
   +2*X13*X23,
  -X1*X2*X23+2*X1*X3+X2^2*X13-X2*X3*X12
   +2*X12*X23-4*X13,
  X1^2*X23-X1*X2*X13-X1*X3*X12+2*X2*X3+2*X12*X13
   -4*X23
]
true
true
true
true
true
true
true
true
true
true
true
true
```

```
true
true
true
true
true
true
true
true
true
true
true
true
true
true
true
true
true
true
true
true
true
(q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4)
(q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4)
(q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4)
(q^2-8*q+16)/(q^2+4*q+4)
(-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4)
(-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4)
(2*q^5-8*q^4-27*q^3+88*q^2+80*q)/
```

```
 (q^3+6*q^2+12*q+8)
[
   (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
   (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
   (q^4-6*q^3+q^2+24*q+16)/(q^2+4*q+4),
   (q^2-8*q+16)/(q^2+4*q+4),
   (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
   (-3*q^3+22*q^2-32*q-32)/(q^2+4*q+4),
   (2*q^5-8*q^4-27*q^3+88*q^2+80*q)/(q^3+6*q^2
     +12*q+8)
]
[
3*q/(q+2),
 -q+2,
 -q+2,
 -2,
 -2,
 (q+8)/(q+2)
]
true
true
true
kaiA32
[
   -2,
   (q-4)/(q-1),
   (q-4)/(q-1),
   (-q+4)/(q-1),
```

```
    (-q+4)/(q-1),
    (-q^2-4*q+14)/(q^2-2*q+1)
]
true
kaiA33
[
    (q^2-2*q-2)/(q-1),
    (-1/2*q^2+q-2)/(q-1),
    (-1/2*q^2+q-2)/(q-1),
    (-5/2*q^2+5*q+2)/(q^2-2*q+1),
    (-5/2*q^2+5*q+2)/(q^2-2*q+1),
    2
]
true
kaiA34
[
    2,
    (-q-2)/(q-1),
    (-q-2)/(q-1),
    (-q-2)/(q-1),
    (-q-2)/(q-1),
    (-q^2+8*q+2)/(q^2-2*q+1)
]
true
true
true
[
    (-2*q^5+16*q^4-32*q^3)/(q^2+4*q+4),
```

```
    (−2*q^5+16*q^4−32*q^3)/(q^2+4*q+4),
    (−2*q^5+16*q^4−32*q^3)/(q^2+4*q+4),
    (−2*q^5+16*q^4−32*q^3)/(q^2+4*q+4),
    (2*q^6−20*q^5+64*q^4−64*q^3)/(q^2+4*q+4),
    (2*q^8−24*q^7+100*q^6−152*q^5+128*q^3)/
      (q^3+6*q^2+12*q+8),
    (2*q^6−20*q^5+64*q^4−64*q^3)/(q^2+4*q+4)
]
kaiB11
[
    −q+2,
    (q^3−4*q^2+2*q+4)/(q+2),
    −q+2,
    −q+2,
    (2*q^2−6*q+4)/(q+2),
    −q+2
]
true
true
kaiB13
[
    −q+2,
    (1/2*q^2+1)/(q+1/2),
    −q+2,
    (−5/2*q+1)/(q+1/2),
    2,
    (−5/2*q+1)/(q+1/2)
]
```

```
true
general:
true
true
true
true
true
true
true
[
    -q+2,
    (-1/3*q+1/3)*X12+(2/3*q^2-7/3*q-4/3)/(q-1),
    (1/3*q-1/3)*X12+(1/3*q^2-2/3*q-8/3)/(q-1),
    X12,
    -X12+(q-4)/(q-1),
    (-4*q^2+11*q+2)/(q^2-2*q+1)
]
true
true
true
true
kaiB31
[
    -q+2,
    -q+2,
    (1/2*q^2+1)/(q+1/2),
    2,
    (-5/2*q+1)/(q+1/2),
```

```
    (−5/2*q+1)/(q+1/2)
]
true
[
    (−2*q^5+16*q^4−32*q^3)/(q^2+4*q+4),
    (−2*q^5+16*q^4−32*q^3)/(q^2+4*q+4),
    (−2*q^5+16*q^4−32*q^3)/(q^2+4*q+4),
    (−2*q^5+16*q^4−32*q^3)/(q^2+4*q+4),
    (2*q^8−24*q^7+100*q^6−152*q^5+128*q^3)/
      (q^3+6*q^2+12*q+8),
    (2*q^6−20*q^5+64*q^4−64*q^3)/(q^2+4*q+4),
    (2*q^6−20*q^5+64*q^4−64*q^3)/(q^2+4*q+4)
]
kaiB33
[
    −q+2,
    −q+2,
    (q^3−4*q^2+2*q+4)/(q+2),
    (2*q^2−6*q+4)/(q+2),
    −q+2,
    −q+2
]
true
true
true
true
true
true
```

```
true
true
[
    (-1/3*q+1/3)*X12+(-1/3*q+1/3)*X13-2/3*q+2/3,
    (1/3*q-1/3)*X13+(1/3*q^2-2/3*q-8/3)/(q-1),
    (1/3*q-1/3)*X12+(1/3*q^2-2/3*q-8/3)/(q-1),
    X12,
    X13,
    -X12-X13+(-3*q^2+6*q+6)/(q^2-2*q+1)
]
true

Total time: 4.440 seconds, Total memory usage: 11.03MB
```