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Nonsymmetric spin models of index 2 on association schemes of small classes

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Abstract

In this paper, we classify nonsymmetric spin models of index 2 on nonsymmetric association schemes of class at most 5.

1 Introduction

Throughout this paper, let X be a non-empty finite set with n elements. We denote by $M_X(\mathbb{C})$ the full matrix ring with complex entries whose rows and columns are indexed by the elements of X. Let $\mathbb{C}^* = \mathbb{C} - \{0\}$. Then $M_X(\mathbb{C}^*)$ is a subset of $M_X(\mathbb{C})$.

1.1 Definitions of spin model and association scheme

A spin model $W \in M_X(\mathbb{C}^*)$ is defined to be a matrix which satisfies two conditions (type II and type III). Whenever we use the symbol $W \in M_X(\mathbb{C}^*)$, the (x, y)-entry of W is denoted by $W(x, y)$ for $x, y \in X$. A *type II*

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matrix on a finite set X is a matrix $W \in M_X(\mathbb{C}^*)$ which satisfies the *type II condition*:

$$\sum_{x \in X} \frac{W(\alpha, x)}{W(\beta, x)} = n\delta_{\alpha, \beta} \qquad \text{(for all } \alpha, \beta \in X\text{)}. \tag{1}$$

Let $W\_ \in M_X(\mathbb{C}^*)$ be defined by $W\_(x, y) = \mathrm{W}((y, x)^{-1}$. Then the type II condition is written as $WW\_ = nI$. Hence, if W is a type II matrix, then W is non-singular with $W^{-1} = n^{-1}W\_$.

A type II matrix $W \in M_X(\mathbb{C}^*)$ is called a *spin model* if W satisfies the *type III condition*:

$$\sum_{x \in X} \frac{W(\alpha, x)W(\beta, x)}{W(\gamma, x)} = D\frac{W(\alpha, \beta)}{W(\alpha, \gamma)W(\gamma, \beta)} \quad \text{(for all } \alpha, \beta, \gamma \in X\text{)} \tag{2}$$

for some nonzero real number D with $D^2 = n$, which is independent of the choice of $\alpha, \beta, \gamma \in X$. It is known that, under the type II condition, (2) is equivalent to the following:

$$\sum_{x \in X} \frac{W(\gamma, x)}{W(\alpha, x)W(\beta, x)} = D\frac{W(\alpha, \gamma)W(\gamma, \beta)}{W(\alpha, \beta)} \quad \text{(for all } \alpha, \beta, \gamma \in X\text{)}. \tag{3}$$

Setting $\beta = \gamma$ in (4),

$$\sum_{x \in X} \frac{1}{W(\alpha, x)} = DW(\beta, \beta). \tag{4}$$

Let R_i $(i = 0, 1, \ldots, d)$ be subsets of $X \times X$ with the property that

(i) $R_0 = \{(x, x) \mid x \in X\}$.

(ii) $X \times X = R_0 \cup \ldots \cup R_d$, $R_i \cap R_j = \emptyset$ if $i \neq j$.

(iii) $R_i^T = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where $R_i^T = \{(x, y) \mid (y, x) \in R_i\}$.

(iv) For $i, j, k \in \{0, 1, \ldots, d\}$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is constant whenever $(x, y) \in R_k$. This constant is denoted by p_{ij}^k.

(v) $p_{ij}^k = p_{ji}^k$ for all i, j, k.

Such a configuration $\mathcal{X}=(X, \{R_i\}_{i=0}^d)$ is called a *commutative association scheme* of class d on X. The non-negative integers p_{ij}^k are called the *intersection numbers* of $\mathcal{X}=(X, \{R_i\}_{i=0}^d)$.

The i-th adjacency matrix $A_i \in M_X(\mathbb{C})$ of $\mathcal{X}=(X, \{R_i\}_{i=0}^d)$ is defined to be the matrix whose rows and columns are indexed by the elements of X and whose (x, y) entries are

$$A_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

A_i is a $(0, 1)$ matrix. The conditions (i), ..., (v) are equivalent to the next (i)′, ..., (iv)′, respectively:

(i)′ $A_0 = I$, the identity matrix.

(ii)′ $A_0 + A_1 + \ldots + A_d = J$, the matrix whose entries are all 1.

(iii)′ $A_i^T = A_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$.

(iv)′ $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$ for all i, j.

(v)′ $A_i A_j = A_j A_i$ for all i, j.

Let $\mathcal{A}$ be the subalgebra of $M_X(\mathbb{C})$ spanned by the adjacency matrices $A_0, A_1, \ldots A_d$. $\mathcal{A}$ is a commutative algebra of $\dim \mathcal{A} = d+1$. $\mathcal{A}$ is called the *Bose-Mesner algebra* of $\mathcal{X}=(X, \{R_i\}_{i=0}^d)$.

Since $\mathcal{A}$ is semi-simple, there uniquely exists the set of the primitive idempotents $\{E_i\}_{i=0}^d$, where $E_0 = \frac{1}{n} J$. So, $\{E_i\}_{i=0}^d$ is the basis of $\mathcal{A}$. Hence, $\mathcal{A}$ has two good basis $\{A_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$. We define the *first eigenmatrix* P of $\mathcal{X}=(X, \{R_i\}_{i=0}^d)$ by the transformation matrix such that

$$(A_0A_1\ldots A_d)=(E_0E_1\ldots E_d)P.$$

Conversely, $\{E_i\}_{i=0}^d$ is expressed by $\{A_i\}_{i=0}^d$ as

$$(E_0E_1\ldots E_d)=\frac{1}{n}(A_0A_1\ldots A_d)Q.$$

Q is called the *second eigenmatrix* of $\mathcal{X}=(X, \{R_i\}_{i=0}^d)$. From these equations, we have

$$PQ=QP=nI.$$

We define the *valency* k_i of R_i and the *multiplicity* m_i of $\mathcal{X}=(X, \{R_i\}_{i=0}^d)$ by

$$k_i=|\{y\in X\,|\,(x, y)\in R_i\}| \quad (x\in X),$$

$$m_i=\dim V_i=\operatorname{rank} E_i,$$

where V_i is the image of $E_i : V \to V_i$. In general, we have

$$k_i=P_{0,i},\ m_i=Q_{0,i}.$$

Latter, we will use the next relations:

$$\frac{Q_{i,j}}{m_j}=\frac{\overline{P_{j,i}}}{k_i}, \tag{5}$$

$$p_{ij}^k=\frac{k_ik_j}{n}\sum_{v=0}^{d}\frac{1}{m_v^2}Q_{i,v}Q_{j,v}\overline{Q_{k,v}}. \tag{6}$$

1.2　Relations between spin models and association schemes

Let $\mathcal{A}$ be the Bose-Mesner algebra of a commutative association scheme $\mathcal{X}=\left(X, \{R_i\}_{i=0}^d\right)$. A *duality* of $\mathcal{A}$ is a linear map $\Psi : \mathcal{A} \to \mathcal{A}$ such that

$$\Psi^2(A)=nA^T \quad \text{for } A\in\mathcal{A}, \tag{7}$$

$$\Psi(AB)=\Psi(A)\circ\Psi(B) \quad \text{for } A, B\in\mathcal{A}. \tag{8}$$

The next theorem is due to [13].

Theorem 1. *Let $W\in M_X(\mathbb{C}^*)$ be a spin model with modulus a. There is a Bose-Mesner algebra $\mathcal{A}$ on X containing W, $W\_$ with duality Ψ given by*

$$\Psi(A)=aW\_^T\circ(W(W\_A)) \tag{9}$$

for all $A \in \mathcal{A}$.

By Theorem 1, a spin model W is expressed by the adjacency matrices of $\mathcal{A}$ as follows:

$$W = \sum_{i=0}^{d} t_i A_i, \tag{10}$$

for some $t_i \in \mathbb{C}^*$ $(i=0, \ldots, d)$. Moreover, it is known that $\mathcal{X} = (X, \{R_i\}_{i=0}^{d})$ with a spin model W is self-dual $(P = \overline{Q})$ using a duality Ψ.

One of the examples of spin models is a Potts model, defined as follows. Let X be a finite set with n elements, and let $I, J \in M_X(\mathbb{C}^*)$ be the identity matrix and the all 1's matrix, respectively. Let u be a complex number satisfying

$$(u^2 + u^{-2})^2 = n \text{ if } n \geq 2,$$
$$u^4 = 1 \text{ if } n = 1. \tag{11}$$

Then a Potts model A_u is defined as

$$A_u = u^3 I - u^{-1}(J - I).$$

As examples of spin models, we know only Potts models [14, 11], spin models on finite abelian groups [3, 6], Jaeger's Higman-Sims model [11], Hadamard models [18, 13], non-symmetric Hadamard models [13], and tensor products of these. Apart from spin models on finite abelian groups, non-symmetric Hadamard models are essentially the only known family of non-symmetric spin models.

If W is a spin model, then by [13, Proposition 2],

$$W^T W^{-1} = A_s, \tag{12}$$

is a permutation matrix. The order of A_s as a permutation is called the *index* of the spin model W. Note that W is symmetric iff $W^T W^{-1} = I$.

A *Hadamard matrix* of order r is a square matrix H of size r with entries ± 1 satisfying $HH^T = rI$. In [13], F. Jaeger and K. Nomura constructed *non-*

symmetric Hadamard models, which are spin models of index 2:

$$W=\begin{pmatrix} \begin{pmatrix}1&1\\1&1\end{pmatrix}\otimes A_u & \begin{pmatrix}1&-1\\-1&1\end{pmatrix}\otimes \xi H \\ \begin{pmatrix}-1&1\\1&-1\end{pmatrix}\otimes \xi H^T & \begin{pmatrix}1&1\\1&1\end{pmatrix}\otimes A_u \end{pmatrix}, \tag{13}$$

where ξ is a primitive 8-th root of unity, $A_u \in M_X(\mathbb{C}^*)$ is a Potts model, and $H \in M_X(\mathbb{C}^*)$ is a Hadamard matrix.

Note that non-symmetric Hadamard models are a modification of the earlier Hadamard models ([13], see also [13, Section 5]), defined by

$$W'=\begin{pmatrix} \begin{pmatrix}1&1\\1&1\end{pmatrix}\otimes A_u & \begin{pmatrix}1&-1\\-1&1\end{pmatrix}\otimes \omega H \\ \begin{pmatrix}1&-1\\-1&1\end{pmatrix}\otimes \omega\xi H^T & \begin{pmatrix}1&1\\1&1\end{pmatrix}\otimes A_u \end{pmatrix}, \tag{14}$$

where ω is a 4-th root of unity.

By [13, Proposition 3] we have the following:

Theorem 2. *Let W be a spin model of index m. Then the following holds:*

(i) *there is a partition of X:*

$$X = X_0 \cup X_1 \cup \ldots \cup X_{m-1} \tag{15}$$

of equal sizes such that

$$W(x, y) = \eta^{i-j} W(y, x) \quad (\forall x \in X_i,\ \forall y \in X_j), \tag{16}$$

where η is a primitive m-th root of unity.

(ii) *Write $W=\sum_{i=0}^{d} t_i A_i$ and $A_s^T = A_{s'}$. Then $t_{s'} = t_0$.*

Now, we fix $p \in X_0$ in (15). Then, we have a disjoint union of X with

$$X = R_0(p) \cup R_1(p) \cup \ldots \cup R_d(p). \tag{17}$$

Since $W^T, W^{-1} \in \mathcal{A}$ by Theorem 1, we have $W^T W^{-1} = A_s \in \mathcal{A}$. Therefore, in (17) there exists $R_s(p)$ such that $|R_s(p)| = 1$.

Lemma 1. *Let $W \in M_X(\mathbb{C}^*)$ be a spin model of index $m \geq 2$. Let $\mathcal{A}$ be the Bose-Mesner algebra such that $W \in \mathcal{A}$ with $\dim\mathcal{A} = d+1$. Then we have*

$$m \leq d+1.$$

Proof. Since the order of A_s is m, A_s^i $(i=0, \ldots, m-1)$ are all distinct. So we have the assertion. □

Lemma 2. *For any $i \in \{0, \ldots, m-1\}$, there exists $j \in \{0, \ldots, m-1\}$ such that*

$$R_i(p) \subset X_j.$$

Proof. For distinct $j_1, j_2 \in \{0, \ldots, m-1\}$, assume that

$$R_i(p) \cap X_{j_1} \neq \emptyset,$$

$$R_i(p) \cap X_{j_2} \neq \emptyset.$$

Then $|j_1 - j_2| \leq m-1$.

Let $x \in R_i(p) \cap X_{j_1}$, $y \in R_i(p) \cap X_{j_2}$. Then (p, x), $(p, y) \in R_i$. From (10), we have

$$t_i = W(p, x) = W(p, y).$$

From (16), we have

$$t_{i'} = \eta^{-j_1} W(x, p) = \eta^{-j_2} W(y, p).$$

Since (x, p), $(y, p) \in R_{i'}$, we have $W(x, p) = W(y, p)$. Therefore we have $\eta^{j_1 - j_2} = 1$. This is a contradiction. □

Lemma 3. *For $j > 0$, R_i with $R_i(p) \subset X_j$ is nonsymmetric.*

Proof. Assume that R_i is symmetric. Let $x \in R_i(p)$. Then

$$(p, x) \in R_i \leftrightarrow (x, p) \in R_i.$$

We have $t_i = W(p, x) = W(x, p)$. On the other hand, by (16)

$$W(p, x) = \eta^{-j} W(x, p).$$

So we have $\eta^{-j} = 1$. This is a contradiction. □

By [13, Proposition 7, Proposition 8], we have the following:

Theorem 3. *The general form of spin models of index 2 is given by*

$$W=\begin{bmatrix} A & A & B & -B \\ A & A & -B & B \\ -B^T & B^T & C & C \\ B^T & -B^T & C & C \end{bmatrix} \quad \textit{with } A,C \textit{ symmetric,} \tag{18}$$

where rows and columns are parameterized by 4 *blocks* Y_1, Y_2, Y_3, Y_4 *of equal sizes as a copy of* Y. *We set* $r=|Y|$. *Then,* $|X|=n=4r$. *Moreover, we have*

$$A_s=\begin{bmatrix} 0 & I & & \\ I & 0 & & \\ & & 0 & I \\ & & I & 0 \end{bmatrix}. \tag{19}$$

$W\in M_X(\mathbb{C}^*)$ is a spin model with loop variable $2D$, where $D^2=r$, if and only if the next (i) and (ii) hold.

(i) A, C are spin models with loop variable D and B is a type II matrix,

(ii) The next identities hold for all $\alpha, \beta, \gamma\in Y$:

$$\sum_{y\in Y}\frac{A(\alpha,y)B(y,\beta)}{B(y,\gamma)}=D\frac{B(\alpha,\beta)}{C(\beta,\gamma)B(\alpha,\gamma)}, \tag{20}$$

$$\sum_{y\in Y}\frac{B(y,\beta)B(y,\gamma)}{A(\alpha,y)}=-D\frac{C(\alpha,\beta)}{B(\alpha,\beta)B(\alpha,\gamma)}. \tag{21}$$

In this paper, we prove the following:

Theorem 4. *Let* $W\in M_X(\mathbb{C}^*)$ *be a spin model of index* 2. *Assume that* W *belongs to the Bose-Mesner algebra of* $\mathcal{X}=\left(X, \{R_i\}_{i=0}^d\right)$ *with at most* $d\leq 5$. *Then,* W *is one of the following:*

(i) *A spin model on the cyclic group of order* 4.

(ii) *Non-symmetric Hadamard models,*

(iii) *In* (18), A *is a spin model on strongly-regular graph,* $B=t^{-1}H$, *where* t *is a primitive* 8*-th root of unity, and* H *is a Hadamard matrix of order* r.

1.3 Spin models of index 2 and association schemes

Throughout this subsection, we consider nonsymmetric spin model of index 2 on nonsymmetric association schemes of class $d \geq 5$.

We decompose $X \times X$ into a diagonal block S_0 and a non-disgonal block S_1 which satisfy Theorem 2(i) as follows:

$$S_0 = (X_0 \times X_0) \cup (X_1 \times X_1),$$
$$S_1 = (X_0 \times X_1) \cup (X_1 \times X_0).$$

By (19), we have

$$A_s \in S_0. \tag{22}$$

For a fixed $p \in X_0$, we have

$$X_0 = S_0(p) \quad (|X_0| = 2r),$$
$$X_1 = S_1(p) \quad (|X_1| = 2r).$$

Lemma 4. *The number of R_i containing in S_1 is even.*

Proof. By Lemma 3,

$$R_i \subset S_1 \longleftrightarrow R_i^T \subset S_1.$$

Hence, S_1 has even relations. □

Lemma 5. *Let R_{i_1}, $R_{i_2} \subset S_0$ and $R_{i_1}^T = R_{i_2}$. Then $t_{i_1} = t_{i_2}$.*

Proof. Let $x \in R_{i_1}(p)$. Then, by the assumption

$$(p, x) \in R_{i_1} \longleftrightarrow (x, p) \in R_{i_2}.$$

Since A, C are symmetric,

$$t_{i_1} = W(p, x) = W(x, p) = t_{i_2}.$$

□

Next, we consider nonsymmetric association schemes of class $d \leq 5$, using Lem-mas 4, 5.

1.3.1 Case of $d = 2$

Let W be a spin model of index 2 on a nonsymmetric association scheme

$\mathcal{X}=(X, \{R_0, R_1, R_2\})$. Then, by Lemma 4 and (22), we have a contradiction.

1.3.2　Case of $d=3$

Let W be a spin model of index 2 on a nonsymmetric association scheme $\mathcal{X}=(X, \{R_0, R_1, R_2, R_3\})$. Then, by suitable rearrangement of indices, by Lemma 4 we may set

$$S_0=R_0\cup R_1,$$
$$S_1=R_2\cup R_3.$$

By (22) we have $k_1=1$. Then $n=4$. In [5], spin models with at most 7 vertices are classified. We know that such a spin model is only the cyclic group of order 4.

1.3.3　Case of $d=4$

Let W be a spin model of index 2 on a nonsymmetric association scheme $\mathcal{X}=(X, \{R_0, R_1, R_2, R_3, R_4\})$. Then, by suitable rearrangement of indices, by Lemma 4 and (22), we may set

$$S_0=R_0\cup R_1\cup R_2 \ (k_2=1)$$
$$S_1=R_3\cup R_4 \ \left(R_3^T=R_4\right).$$

Then

$$k_1=2r-1,$$
$$k_3=k_4=r.$$

Since A in (18) takes the same non-diagonal entries, A is a Potts model. Since B is a type II matrix, we have $t_4=-t_3$. We set $B=t_3^{-1}H$, where H is a Hadamard matrix. So, W is a nonsymmetric Hadamard model.

1.3.4 Case of $d=5$

Let W be a spin model of index 2 on a nonsymmetric association scheme $\mathcal{X}=(X, \{R_0, R_1, R_2, R_3, R_4, R_5\})$. Then, by suitable rearrangement of indices, by Lemma 4 and (22), we have the next two possibilities:

$$\begin{cases} S_0=R_0\cup R_1(k_1=1), \\ S_1=R_2\cup R_3\cup R_4\cup R_5. \end{cases}$$

$$\begin{cases} S_0=R_0\cup R_1\cup R_2\cup R_3(k_3=1), \\ S_1=R_4\cup R_5\left(R_4^T=R_5\right). \end{cases}$$

The former leads us to a contradiction by $|S_0(p)|=2$ and $|S_1(p)|\geq 4$.

We consider the latter. If $R_1^T=R_2$, then by Lemma 5 we have $t_1=t_2$. By $R_1\cup R_2$, this case is reduced to $d=4$. Similarly, if R_1, R_2 are symmetric and $t_1=t_2$, then $R_1\cup R_2$ is reduced to $d=4$. Therefore, we assume that R_1, R_2 are symmetric and $t_1\neq t_2$.

In what follows, we are mainly interested in the latter case, i.e.,

$$\begin{cases} X\times X=S_0\cup S_1, \\ S_0=R_0\cup R_1\cup R_2\cup R_3(R_1, R_2 : \text{symmetric}, k_3=1), \\ S_1=R_4\cup R_5\left(R_4^T=R_5\right), \\ W=\sum_{i=0}^{5} t_iA_i(t_3=t_0,\ t_1\neq t_2). \end{cases} \tag{23}$$

Then we want to determine the general form of the adjacency matrices $\{A_i\}_{i=0}^5$ with (23). Before that, we mention the general facts of (23).

Let W be a spin model of index 2. Assume that W belongs to the Bose-Mesner algebra $\mathcal{A}=\langle A_0, A_1, \ldots, A_d\rangle$ with the next condition:

$$
\begin{cases}
X\times X=S_0\cup S_1,\\
S_0=\cup_{i=0}^{d-2}R_i(R_i:\text{symmetric}, k_{d-2}=1),\\
S_1=R_{d-1}\cup R_d\left(R_{d-1}^T=R_d\right),\\
W=\sum_{i=0}^{d}t_iA_i(t_{d-2}=t_0, t_1, \ldots, t_{d-3}:\text{distinct}),
\end{cases} \tag{24}
$$

where A_{d-2} is given by the form (19). Then we have the following:

Lemma 6. *Let* W *be a spin model of index* 2 *on a nonsymmetric association schemes of class* d *with the condition* (24). *Then we have*

$$t_{d-1}=-t_d.$$

Proof. Since B is a type II matrix, B is covered by distinct values t_{d-1}, t_d. By $S_1=R_{d-1}\cup R_d$, we set $B=t_{d-1}H$, H is a Hadamard matrix. Hence, $t_{d-1}=-t_d$. □

Lemma 7. *The adjacency matrices* A_{d-1}, A_d *are given by*

$$
A_{d-1}=\begin{bmatrix}
 & & \frac{J+H}{2} & \frac{J-H}{2}\\
 & & \frac{J-H}{2} & \frac{J+H}{2}\\
\frac{J-H^T}{2} & \frac{J+H^T}{2} & & \\
\frac{J+H^T}{2} & \frac{J-H^T}{2} & &
\end{bmatrix}, \tag{25}
$$

$$
A_d=\begin{bmatrix}
 & & \frac{J-H}{2} & \frac{J+H}{2}\\
 & & \frac{J+H}{2} & \frac{J-H}{2}\\
\frac{J+H^T}{2} & \frac{J-H^T}{2} & & \\
\frac{J-H^T}{2} & \frac{J+H^T}{2} & &
\end{bmatrix}=A_{d-1}^T. \tag{26}
$$

Proof. It is immediate from (18) and Lemma 6.

Lemma 8. *For* $i \in \{1, \ldots, d-3\}$ *we have*

$$A_i A_{d-2} = A_i.$$

Proof. By suitable rearrangement of indices, assume that

$$\begin{aligned} A_1 A_{d-2} &= A_2, \\ A_3 A_{d-2} &= A_4, \\ &\vdots \\ A_{2\ell+1} A_{d-2} &= A_{2\ell+2} (2\ell+2 \leq d-3). \end{aligned}$$

Then, using $A_{d-2}^2 = I$, we have

$$\begin{aligned} A_2 A_{d-2} &= A_1, \\ A_4 A_{d-2} &= A_3, \\ &\vdots \\ A_{2\ell+2} A_{d-2} &= A_{2\ell+1}. \end{aligned}$$

From (24) and Lemma 6

$$\begin{aligned} A_{d-2} W &= t_0 A_{d-2} + \sum_{i=1}^{d-3} t_i A_{d-2} A_i + t_0 A_0 + t_{d-1} A_{d-2} A_{d-1} - t_{d-1} A_{d-2} A_d \\ &= t_0 A_{d-2} + (t_2 A_1 + t_1 A_2) + \ldots + (t_{2\ell+2} A_{2\ell+1} + t_{2\ell+1} A_{2\ell+2}) \\ &\quad + t_0 A_0 + t_{d-1} A_d - t_{d-1} A_{d-1}, \end{aligned}$$

$$W^T = t_0 A_0 + \sum_{i=1}^{d-3} t_i A_i + t_0 A_{d-2} + t_{d-1} A_d - t_d A_{d-1}.$$

From (16)

$$\begin{aligned} W^T = A_{d-2} W \longleftrightarrow (t_1 - t_2)(A_1 - A_2) + \ldots + (t_{2\ell+1} - t_{2\ell+2})(A_{2\ell+1} - A_{2\ell+2}) \\ = 0. \end{aligned}$$

Since $A_1, \ldots, A_{2\ell+2}$ are all distinct, we have $t_1 = t_2, \ldots, t_{2\ell+1} = t_{2\ell+2}$. This is a contradiction by Lemma 5. □

Lemma 9. *Let W be a spin model of index 2 on a nonsymmetric association schemes of class d with the condition* (24). *Then,* t_4 *is a primitive 8-th root of* *unity.*

Proof. Putting $\beta=\gamma$ in (21), we have

$$\begin{aligned}
\mathrm{LHS} &= \sum_{y\in Y}\frac{B(y,\beta)^2}{A(\alpha,y)} \\
&= \sum_{y\in Y}\frac{t_4^2}{A(\alpha,y)} \\
&= t_4^2\sum_{y\in Y}\frac{1}{A(\alpha,y)} \\
&= Dt_0t_4^2. \\
\mathrm{RHS} &= -D\frac{C(\beta,\beta)}{B(\alpha,\beta)^2} \\
&= -D\frac{t_0}{t_4^2} \qquad \text{(by (4))}.
\end{aligned}$$

Hence, we have $t_4^4=-1$. □

Next, we determine the general form of $A_i(i=1,\ldots,d-3)$. Then we have the following:

Lemma 10. *The adjacency matrices* $A_i(i=1,\ldots,d-3)$ *of a nonsymmetric associ-ation scheme of class d with the condition* (24). *are given by*

$$A_i=\begin{bmatrix} C_i & C_i & & \\ C_i & C_i & & \\ & & F_i & F_i \\ & & F_i & F_i \end{bmatrix},$$

where C_i, F_i *are symmetric.*

Proof. Since A_i is symmetric, we firstly consider the next two cases:

$$A_i=\begin{bmatrix} 0 & C & & \\ C^T & 0 & & \\ & & 0 & F \\ & & F^T & 0 \end{bmatrix} \text{ or } A_i=\begin{bmatrix} C_1 & 0 & & \\ 0 & C_2 & & \\ & & F_1 & 0 \\ & & 0 & F_2 \end{bmatrix},$$

where C_1, C_2, F_1, F_2 are symmetric. However, the both cases do not satisfy $A_iA_{d-3}=A_i$. By Lemma 8, this is a contradiction. Therefore, we set

$$A_i=\begin{bmatrix} C_1 & C_2 & & \\ C_2^T & C_3 & & \\ & & F_1 & F_2 \\ & & F_2^T & F_3 \end{bmatrix},$$

where C_1, C_3, F_1, F_3 are symmetric. Then

$$A_iA_{d-3}=\begin{bmatrix} C_1 & C_2 & & \\ C_2^T & C_3 & & \\ & & F_1 & F_2 \\ & & F_2^T & F_3 \end{bmatrix}\begin{bmatrix} 0 & I & & \\ I & 0 & & \\ & & 0 & I \\ & & I & 0 \end{bmatrix}=\begin{bmatrix} C_2 & C_1 & & \\ C_3 & C_2^T & & \\ & & F_2 & F_1 \\ & & F_3 & F_2^T \end{bmatrix}=A_i,$$

$$A_{d-3}A_i=\begin{bmatrix} 0 & I & & \\ I & 0 & & \\ & & 0 & I \\ & & I & 0 \end{bmatrix}\begin{bmatrix} C_1 & C_2 & & \\ C_2^T & C_3 & & \\ & & F_1 & F_2 \\ & & F_2^T & F_3 \end{bmatrix}=\begin{bmatrix} C_2^T & C_3 & & \\ C_1 & C_2 & & \\ & & F_2^T & F_3 \\ & & F_1 & F_2 \end{bmatrix}=A_i.$$

From these, we have

$$C_2=C_2^T,$$

$$C_1=C_3,$$

$$C_1=C_2,$$

$$C_2^T=C_3.$$

Hence, we have the assertion. □

Here, we return to the case (23). Let W be a spin model of index 2 on a nonsym-metric association scheme of class 5 with the condition (23). Then, by Lemmas 7, 10, $\{A_i\}_{i=0}^5$ are given by

$$A_0=\begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix},$$

$$A_1=\begin{bmatrix} C_1 & C_1 & & \\ C_1 & C_1 & & \\ & & F_1 & F_1 \\ & & F_1 & F_1 \end{bmatrix} \quad (C_1, F_1 \text{ : symmetric}),$$

$$A_2=\begin{bmatrix} C_2 & C_2 & & \\ C_2 & C_2 & & \\ & & F_2 & F_2 \\ & & F_2 & F_2 \end{bmatrix} \quad (C_2, F_2 \text{ : symmetric}),$$

$$A_3=\begin{bmatrix} 0 & I & & \\ I & 0 & & \\ & & 0 & I \\ & & I & 0 \end{bmatrix},$$

$$A_4=\begin{bmatrix} & & \frac{J+H}{2} & \frac{J-H}{2} \\ & & \frac{J-H}{2} & \frac{J+H}{2} \\ \frac{J-H^T}{2} & \frac{J+H^T}{2} & & \\ \frac{J+H^T}{2} & \frac{J-H^T}{2} & & \end{bmatrix},$$

$$A_5=\begin{bmatrix} & & \frac{J-H}{2} & \frac{J+H}{2} \\ & & \frac{J+H}{2} & \frac{J-H}{2} \\ \frac{J+H^T}{2} & \frac{J-H^T}{2} & & \\ \frac{J-H^T}{2} & \frac{J+H^T}{2} & & \end{bmatrix}=A_4^T.$$

Then, W with the condition (23) is given by

$$W=t_0A_0+t_1A_1+t_2A_2+t_0A_3+t_4A_4-t_4A_5 \quad \left(t_4^4=-1\right).$$

In (18), A is symmetric spin model. From the shape of $\{A_i\}_{i=0}^d$, we have

$$A=t_0I+t_1C_1+t_2C_2,$$

and $\mathcal{A}=\langle I, C_1, C_2\rangle$ is the Bose-Mesner algebra of a strongly regular graph. Then we have the next first eigenmatrix $\tilde{P}$ of $\mathcal{Y}=(Y, \{I, C_1, C_2\})$ given by t_0, t_1, t_2 as follows:

Lemma 11. *Let A be a symmetric spin model on a strongly regular graph $\mathcal{Y}=(Y, \{I, C_1, C_2\})$, where I, C_1, $C_2\in M_Y(\mathbb{C})$ are adjacency matrices. Let $A=t_0I+t_1C_1+t_2C_2$, where t_0, t_1, t_2 are nonzero complex numbers. Then, the first eigenmatrix $\tilde{P}$ of $\mathcal{Y}$ is given by*

$$\tilde{P}=\begin{bmatrix} 1 & \frac{t_1(\epsilon t_0^2+1)(\epsilon-t_0t_1)(t_0+\epsilon t_1^3)}{t_0^2(t_1^4-1)(t_1^2+\epsilon)} & \frac{t_1(t_0^2+\epsilon)(t_0t_1^3+1)(t_0-t_1)}{t_0^2(t_1^4-1)(t_1^2+\epsilon)} \\ 1 & \frac{-t_1(t_0^2t_1^2-1)}{t_0(t_1^4-1)} & \frac{(t_0t_1^3+1)(t_0-t_1)}{t_0(t_1^4-1)} \\ 1 & -\frac{\epsilon t_0^2t_1+t_0(t_1^4-1)-et_1^3}{t_0(t_1^4-1)} & \frac{\epsilon t_1(t_0^2-t_1^2)}{t_0(t_1^4-1)} \end{bmatrix},$$

$$t_1t_2=\epsilon\in\{1, -1\},$$

$$D'=\frac{(t_0-t_1)(t_0t_1-\epsilon)}{\epsilon t_0(t_1^2+\epsilon)},\ D'^2=r.$$

Moreover, $\tilde{P}$ is self-dual.

Proof. The proof basically depends on the method of K. Nomura. Let

$$P=\begin{bmatrix}1 & x_0 & y_0\\ 1 & x_1 & y_1\\ 1 & x_2 & y_2\end{bmatrix}.$$

We have

$$\sum_{i=0}^{2} p_{ij}=r\delta_{i,0},$$

$$\sum_{j=0}^{2} p_{i,j}t_j=D't_i^{-1},$$

$$\sum_{j=0}^{2} p_{i,j}t_j^{-1}=D't_i,$$

where $r=D'^2$. For $i\in\{1, 2\}$ we have

$$1+x_i+y_i=0,$$

$$\left(t_0-\frac{D'}{t_i}\right)+t_1x_i+t_2y_i=0,$$

$$\left(\frac{1}{t_0}-D't_i\right)+\frac{x_i}{t_1}+\frac{y_i}{t_2}=0.$$

From these we have

$$\begin{bmatrix}1 & 1 & 1\\ t_0-\frac{D'}{t_i} & t_1 & t_2\\ \frac{1}{t_0}-D't_i & \frac{1}{t_1} & \frac{1}{t_2}\end{bmatrix}\begin{bmatrix}1\\ x_i\\ y_i\end{bmatrix}=0.$$

We set

$$H=\begin{bmatrix}1 & 1 & 1\\ t_0-\frac{D'}{t_i} & t_1 & t_2\\ \frac{1}{t_0}-D't & \frac{1}{t_1} & \frac{1}{t_2}\end{bmatrix}.$$

Then

H has non-trivial solutions $\Longleftrightarrow$ $\det H=0$.

$$\begin{aligned}\det H&=t_0t_1t_2(t_1-t_2)D't^2+(t_0-t_1)(t_1-t_2)(t_2-t_0)t+t_0(t_1-t_2)D'\\&=t_0t_1t_2(t_1-t_2)D'(t-t_1)(t-t_2)\\&=t_0t_1t_2(t_1-t_2)D't^2-t_0t_1t_2(t_1-t_2)D'(t_1+t_2)t\\&\quad+t_0t_1t_2(t_1-t_2)D't_1t_2.\end{aligned}$$

By Newton's relations, we have

$$(t_0-t_1)(t_1-t_2)(t_2-t_0)=-t_0t_1t_2(t_1-t_2)D'(t_1+t_2),$$

$$t_0(t_1-t_2)D'=t_0t_1t_2(t_1-t_2)D't_1t_2.$$

From the second equation, we have

$$t_1t_2=\epsilon\in\{1,-1\}.$$

From the first equation, we have

$$D'=\frac{(t_0-t_1)(t_0t_1-\epsilon)}{\epsilon t_0(t_1^2+\epsilon)}.$$

□

Using $\tilde{P}$ in Lemma 11, by (6) we calculate the intersection numbers p_{ij}^k as follows:

Lemma 12. *The intersection numbers* $\tilde{p}_{ij}^k(i,j,k=0,1,2)$ *of* $\mathcal{Y}=(Y,\{I,C_1,C_2\})$ *with the first eigenmatrix* $\tilde{P}$ *are given by the following:*

$$p_{00}^{\tilde{0}}=1,$$

$$p_{00}^{\tilde{1}}=1,$$

$$p_{00}^{\tilde{2}}=1,$$

$$p_{11}^{\tilde{0}}=\frac{t_1(\epsilon t_0^2+1)(\epsilon-t_0t_1)(t_0+\epsilon t_1^3)}{t_0^2(t_1^4-1)(t_1^2+\epsilon)},$$

$$p_{11}^{\tilde{1}}=\frac{(t_0+t_1)(t_0t_1-1)(t_0t_1+1)(t_0-t_1^5)}{t_0^2(t_1^4-1)^2},$$

$$p_{11}^{\tilde{2}}=\frac{t_1(\epsilon-t_0t_1)(t_0+\epsilon t_1^3)(t_1^2-t_0^2)}{t_0^2(t_1^4-1)^2},$$

$$p_{12}^{\tilde{1}}=-\frac{\epsilon t_1(t_0+t_1)(t_0-t_1)^2(t_0t_1^3+1)}{t_0^2(t_1^4-1)^2},$$

$$p_{12}^{\tilde{2}}=-\frac{\epsilon t_1(t_0+\epsilon t_1^3)(t_0t_1+\epsilon)(t_0t_1-\epsilon)^2}{t_0^2(t_1^4-1)^2},$$

$$p_{22}^{\tilde{0}}=\frac{t_1(t_0^2+\epsilon)(t_0t_1^3+1)(t_0-t_1)}{t_0^2(t_1^4-1)(t_1^2+\epsilon)},$$

$$p_{22}^{\tilde{1}}=\frac{t_1(t_0-t_1)(t_0t_1+1)(t_0t_1-1)(t_0t_1^3+1)}{t_0^2(t_1^4-1)^2},$$

$$p_{22}^{\tilde{2}}=\frac{(t_0-t_1)(t_0+t_1)(t_0t_1+\epsilon)(t_0t_1^5-\epsilon)}{t_0^2(t_1^4-1)^2}.$$

Proof. The proof is based on the Mathematical software “Maple”:

```
restart;
with(LinearAlgebra): interface(rtablesize＝infinity):
P :＝ Matrix( [[1,t1*(e*t0^2+1)*(-t0*t1+e)*(t1^3+e*t0)
/(e*t0^2*(t1^4-1)*(t1^2+e)),
-t1*(t0^2+e)*(t0*t1^3+1)*(t1-t0)/(t0^2*(t1^4-1)*(t1^2+e))],
[1,-t1*(t0^2*t1^2-1)/(t0*(t1^4-1)),(t0*t1^3+1)*(t0-t1)/
(t0*(t1^4-1))],
[1,-(e*t0^2*t1+t0*(t1^4-1)-e*t1^3)/(t0*(t1^4-1))
,e*t1*(t0^2-t1^2)/(t0*(t1^4-1))]] );
t2:＝e/t1;E:＝(t0-t1)*(t0*t1-e)/(e*t0*(t1^2+e));
e:＝-1;
for i1 from 1 to 3 do
for i2 from 1 to 3 do
for i3 from 1 to 3 do
p[i1-1,i2-1,i3-1]:＝( P[1,i1]*P[1,i2]/(E^2) )
```

```
*add((1/(P[1,y]^2))*P[i1,y]*P[i2,y]*P[i3,y],y=1..3);
od; od; od;
for i1 from 1 to 3 do
for i2 from 1 to 3 do
for i3 from 1 to 3 do
print( [i1-1,i2-1,i3-1], factor(p[i1-1,i2-1,i3-1]) );
od; od; od;
```

□

Using Lemma 12, we determine the intersection numbers $p_{ij}^k (i, j, k=0, \dots, 5)$ of a nonsymmetric association scheme with (23). We set

$$k_1 = 2\tilde{k}_1 = 2p_{11}^{\tilde{0}},$$
$$k_2 = 2\tilde{k}_2 = 2p_{22}^{\tilde{0}},$$
$$p_{11}^1 = 2p_{11}^{\tilde{1}},$$
$$p_{11}^2 = 2p_{11}^{\tilde{2}},$$
$$p_{22}^1 = 2p_{22}^{\tilde{1}},$$
$$p_{22}^2 = 2p_{22}^{\tilde{2}}.$$

Then, we have the following:

Lemma 13. *The intersection numbers* $p_{ij}^k (i, j, k=0, \dots, 5)$ *of a nonsymmetric asso-ciation scheme with* (23) *are given by the following:*

$$A_1^2 = 2\tilde{k}_1 A_0 + 2p_{11}^{\tilde{1}} A_1 + 2p_{11}^{\tilde{2}} A_2 + 2\tilde{k}_1 A_3,$$
$$A_1 A_2 = 2p_{12}^{\tilde{1}} A_1 + 2p_{12}^{\tilde{2}} A_2,$$
$$A_1 A_3 = A_1,$$
$$A_1 A_4 = \tilde{k}_1 (A_4 + A_5),$$
$$A_1 A_5 = \tilde{k}_1 (A_4 + A_5),$$
$$A_2^2 = 2\tilde{k}_2 A_0 + 2p_{22}^{\tilde{1}} A_1 + 2p_{22}^{\tilde{2}} A_2 + 2\tilde{k}_2 A_3,$$
$$A_2 A_3 = A_2,$$

$$A_2A_4=\tilde{k}_2(A_4+A_5),$$

$$A_2A_5=\tilde{k}_2(A_4+A_5),$$

$$A_3^2=A_0,$$

$$A_3A_4=A_5,$$

$$A_3A_5=A_4,$$

$$A_4^2=\frac{r}{2}(A_1+A_2+2A_3),$$

$$A_4A_5=\frac{r}{2}(2A_0+A_1+A_2),$$

$$A_5^2=\frac{r}{2}(A_1+A_2+2A_3).$$

Proof.

$$A_1^2=\begin{bmatrix} C_1 & C_1 & & \\ C_1 & C_1 & & \\ & & F_1 & F_1 \\ & & F_1 & F_1 \end{bmatrix}\begin{bmatrix} C_1 & C_1 & & \\ C_1 & C_1 & & \\ & & F_1 & F_1 \\ & & F_1 & F_1 \end{bmatrix}=2\begin{bmatrix} C_1^2 & C_1^2 & & \\ C_1^2 & C_1^2 & & \\ & & F_1^2 & F_1^2 \\ & & F_1^2 & F_1^2 \end{bmatrix}$$

$$=2(\tilde{k}_1A_0+p_{11}^{\tilde{1}}A_1+p_{11}^{\tilde{2}}A_2+\tilde{k}_1A_3).$$

$$A_1A_2=\begin{bmatrix} C_1 & C_1 & & \\ C_1 & C_1 & & \\ & & F_1 & F_1 \\ & & F_1 & F_1 \end{bmatrix}\begin{bmatrix} C_2 & C_2 & & \\ C_2 & C_2 & & \\ & & F_2 & F_2 \\ & & F_2 & F_2 \end{bmatrix}$$

$$=2\begin{bmatrix} C_1C_2 & C_1C_2 & & \\ C_1C_2 & C_1C_2 & & \\ & & F_1F_2 & F_1F_2 \\ & & F_1F_2 & F_1F_2 \end{bmatrix}$$

$$=2(p_{12}^{\tilde{1}}A_1+p_{12}^{\tilde{2}}A_2).$$

$A_1A_3=A_1$ (by Lemma 8),

$$A_1A_4=\begin{bmatrix} C_1 & C_1 & & \\ C_1 & C_1 & & \\ & & F_1 & F_1 \\ & & F_1 & F_1 \end{bmatrix}\begin{bmatrix} & & \frac{J+H}{2} & \frac{J-H}{2} \\ & & \frac{J-H}{2} & \frac{J+H}{2} \\ \frac{J-H^T}{2} & \frac{J+H^T}{2} & & \\ \frac{J+H^T}{2} & \frac{J-H^T}{2} & & \end{bmatrix}$$

$$=\begin{bmatrix} & & C_1\left(\frac{J+H}{2}+\frac{J-H}{2}\right) & C_1\left(\frac{J-H}{2}+\frac{J+H}{2}\right) \\ & & C_1\left(\frac{J+H}{2}+\frac{J-H}{2}\right) & C_1\left(\frac{J-H}{2}+\frac{J+H}{2}\right) \\ F_1\left(\frac{J-H^T}{2}+\frac{J+H^T}{2}\right) & F_1\left(\frac{J+H^T}{2}+\frac{J-H^T}{2}\right) & & \\ F_1\left(\frac{J-H^T}{2}+\frac{J+H^T}{2}\right) & F_1\left(\frac{J+H^T}{2}+\frac{J-H^T}{2}\right) & & \end{bmatrix}$$

$$=\begin{bmatrix} & & C_1J & C_1J \\ & & C_1J & C_1J \\ F_1J & F_1J & & \\ F_1J & F_1J & & \end{bmatrix}=\tilde{k}_1\begin{bmatrix} & & J & J \\ & & J & J \\ J & J & & \\ J & J & & \end{bmatrix}$$

$$=\tilde{k}_1(A_4+A_5).$$

$$A_1A_5=\frac{k_1}{2}(A_4+A_5)\text{(similar to } A_1A_5\text{)},$$

$$A_2^2=2\tilde{k}_2A_0+2p^{\tilde{1}}_{22}A_1+2p^{\tilde{2}}_{22}A_2+2\tilde{k}_2A_3\text{(similar to } A_1^2\text{)},$$

$$A_2A_3=A_2\text{(by Lemma 8)},$$

$$A_2A_4=\tilde{k}_2(A_4+A_5)\text{(similar to } A_1A_4\text{)},$$

$$A_2A_5=\tilde{k}_2(A_4+A_5)\text{(similar to } A_2A_4\text{)},$$

$$A_4^2=\frac{r}{2}(A_1+A_2+2A_3)\text{(by [13, p. 264])},$$

$$A_4A_5=\frac{r}{2}(2A_0+A_1+A_2)\text{(by [13, p. 264])},$$

$$A_5^2=\frac{r}{2}(A_1+A_2+2A_3)\text{(by [13, p. 264])}.$$

Lemma 14. *The Bose-Mesner algebra* $\mathcal{A}$ *with* (23) *has a duality*

$$\Psi(A)=t_0W\_^T\circ(W(W\_\circ A))$$

for all $M\in\mathcal{A}$*. The matrix of* Ψ *in the basis* $\{A_i|i=0, \ldots, 5\}$ *is*

$$P=\begin{bmatrix}1 & k_1 & k_2 & 1 & r & r\\ 1 & p_{11} & p_{12} & 1 & 0 & 0\\ 1 & p_{21} & p_{22} & 1 & 0 & 0\\ 1 & k_1 & k_2 & 1 & -r & -r\\ 1 & 0 & 0 & -1 & p_{44} & -p_{44}\\ 1 & 0 & 0 & -1 & -p_{44} & p_{44}\end{bmatrix},$$

where

$$k_1=\frac{2t_1(\epsilon t_0^2+1)(\epsilon-t_0t_1)(t_0+\epsilon t_1^3)}{t_0^2(t_1^4-1)(t_1^2+\epsilon)},$$

$$k_2=\frac{2t_1(t_0^2+\epsilon)(t_0t_1^3+1)(t_0+t_1)}{t_0^2(t_1^4-1)(t_1^2+\epsilon)},$$

$$p_{11}=-\frac{2t_1(t_0t_1-1)(t_0t_1+1)}{t_0(t_1^4-1)},$$

$$p_{12}=\frac{2(t_0t_1^3+1)(t_0-t_1)}{t_0(t_1^4-1)},$$

$$p_{21}=\frac{2(\epsilon t_0+t_1^3)(\epsilon-t0t_1}{t_0(t_1^4-1)},$$

$$p_{22}=\frac{2\epsilon t_1(t_0-t_1)(t_0+t_1)}{t_0\left(t_1^4-1\right)},$$

$$p_{44}=\frac{(t_0t_1+1)(t_0-t_1)}{t_0(t_1^2-1)}\sqrt{-1}.$$

Proof. Let

$$W=t_0A_0+t_1A_1+t_2A_2+t_0A_3+t_4A_4+t_4A_5.$$

Then we have

$$\begin{aligned}W\_&=t_0^{-1}A_0+t_1^{-1}A_1+t_2^{-1}A_2+t_0^{-1}A_3+t_4^{-1}(A_4^T-A_5^T)\\&=t_0^{-1}A_0+t_1^{-1}A_1+t_2^{-1}A_2+t_0^{-1}A_3+t_4^{-1}(A_5-A_4),\end{aligned}$$

$$W\_^T=t_0^{-1}A_0+t_1^{-1}A_1+t_2^{-1}A_2+t_0^{-1}A_3+t_4^{-1}A_4-t_4^{-1}A_5.$$

$\underline{\Psi(A_1)=t_0W\_^T\circ(W(W\_\circ A_1))}$:

$$W\_\circ A_1=t_1^{-1}A_1,$$

$$\begin{aligned}W(W\_\circ A_1)&=t_1^{-1}(t_0A_0+t_1A_1+t_2A_2+t_0A_3+t_4A_4-t_4A_5)A_1\\&=t_0t_1^{-1}A_1+A_1^2+t_1^{-1}t_2A_1A_2+t_0t_1^{-1}A_1A_3+t_1^{-1}t_4A_1A_4\\&\quad-t_1^{-1}t_4A_1A_5\\&=k_1A_0+\left(2\frac{t_0}{t_1}+\frac{t_2}{t_1}p_{12}^1+p_{11}^1\right)A_1+\left(\frac{t_2}{t_1}p_{12}^2+p_{11}^2\right)A_2+k_1A_3.\end{aligned}$$

$$\begin{aligned}t_0W\_^T\circ(W(W\_\circ A_1))&=(A_0+t_0t_1^{-1}A_1+t_0t_2^{-1}A_2+A_3+t_0t_4^{-1}A_4-t_0t_4^{-1}A_5)\\&\quad\circ\left(k_1A_0+\left(2\frac{t_0}{t_1}+\frac{t_2}{t_1}p_{12}^1+p_{11}^1\right)A_1\right.\\&\quad\left.+\left(\frac{t_2}{t_1}p_{12}^2+p_{11}^2\right)A_2+k_1A_3\right)\\&=k_1A_0+\frac{t_0}{t_1}\left(2\frac{t_0}{t_1}+\frac{t_2}{t_1}p_{12}^2+p_{11}^1\right)A_1\\&\quad+\frac{t_0}{t_2}\left(\frac{t_2}{t_1}p_{12}^2+p_{11}^2\right)A_2+k_1A_3.\end{aligned}$$

By Lemma 12,

$$\frac{t_0}{t_1}\left(2\frac{t_0}{t_1}+\frac{t_2}{t_1}p_{12}^2+p_{11}^1\right)=-\frac{2t_1(t_0t_1-1)(t_0t_1+1)}{t_0(t_1^4-1)},$$

$$\frac{t_0}{t_2}\left(\frac{t_2}{t_1}p_{12}^2+p_{11}^2\right)=-\frac{2(\epsilon t_0+t_1^3)(\epsilon-t0t_1}{t_0(t_1^4-1)}.$$

$\underline{\Psi(A_2)=t_0W\_^T\circ(W(W\_\circ A_2))}$:

$$W\_\circ A_2=t_2^{-1}A_2,$$

$$\begin{aligned}W(W\_\circ A_2)&=t_2^{-1}(t_0A_0+t_1A_1+t_2A_2+t_0A_3+t_4A_4-t_4A_5)A_2\\&=t_0t_2^{-1}A_1+t_1t_2^{-1}A_1A_2+A_2^2+t_0t_2^{-1}A_2A_3+t_2^{-1}t_4A_2A_4\\&\quad -t_2^{-1}t_4A_2A_5\\&=k_2A_0+\left(\frac{t_1}{t_2}p_{12}^1+p_{22}^1\right)A_1+\left(2\frac{t_0}{t_2}+\frac{t_1}{t_2}p_{12}^2+p_{22}^2\right)A_2+k_2A_3.\end{aligned}$$

$$\begin{aligned}t_0W\_^T\circ(W(W\_\circ A_2))&=(A_0+t_0t_1^{-1}A_1+t_0t_2^{-1}A_2+A_3+t_0t_4^{-1}A_4-t_0t_4^{-1}A_5)\\&\quad\circ\left(k_2A_0+\left(\frac{t_1}{t_2}p_{12}^1+p_{22}^1\right)A_1\right.\\&\quad\left.+\left(2\frac{t_0}{t_2}+\frac{t_1}{t_2}p_{12}^2+p_{22}^2\right)A_2+k_2A_3\right)\\&=k_2A_0+\frac{t_0}{t_1}\left(\frac{t_1}{t_2}p_{12}^1+p_{22}^1\right)A_1\\&\quad+\frac{t_0}{t_2}\left(2\frac{t_0}{t_2}+\frac{t_1}{t_2}p_{12}^2+p_{22}^2\right)A_2+k_2A_3.\end{aligned}$$

By Lemma 12,

$$\frac{t_0}{t_1}\left(\frac{t_1}{t_2}p_{12}^1+p_{22}^1\right)=\frac{2(t_0t_1^3+1)(t_0-t_1)}{t_0(t_1^4-1)},$$

$$\frac{t_0}{t_2}\left(2\frac{t_0}{t_2}+\frac{t_1}{t_2}p_{12}^2+p_{22}^2\right)=\frac{2\epsilon t_1(t_0-t_1)(t_0+t_1)}{t_0(t_1^4-1)}.$$

$\underline{\Psi(A_3)=t_0W\_^T\circ(W(W\_\circ A_3))}$:

$$W\_\circ A_3=t_0^{-1}A_3,$$

$$
\begin{aligned}
W(W_{-}\circ A_3) &= t_0^{-1}(t_0A_0+t_1A_1+t_2A_2+t_0A_3+t_4A_4-t_4A_5)A_3 \\
&= A_3+t_1t_0^{-1}A_1A_3+t_2t_0^{-1}A_2A_3+A_3^2+t_4t_0^{-1}A_3A_4 \\
&\quad -t_4t_0^{-1}A_3A_5 \\
&= A_3+\frac{t_1}{t_0}A_1+\frac{t_2}{t_0}A_2+A_0+\frac{t_4}{t_0}A_5-\frac{t_4}{t_0}A_4.
\end{aligned}
$$

$$
\begin{aligned}
t_0W_{-}^{T}\circ(W(W_{-}\circ A_3)) &= (A_0+t_0t_1^{-1}A_1+t_0t_2^{-1}A_2+A_3+t_0t_4^{-1}A_4-t_0t_4^{-1}A_5) \\
&\quad \circ\left(A_0+\frac{t_1}{t_0}A_1+\frac{t_2}{t_0}A_2+A_3+\frac{t_4}{t_0}A_5-\frac{t_4}{t_0}A_4\right) \\
&= A_0+A_1+A_2+A_3-A_4-A_5.
\end{aligned}
$$

<u>$\Psi(A_4)=t_0W_{-}^{T}\circ(W(W_{-}\circ A_4))$</u>:

$$
\begin{aligned}
W_{-}\circ A_4 &= -t_4^{-1}A_4, \\
W(W_{-}\circ A_4) &= -t_4^{-1}(t_0A_0+t_1A_1+t_2A_2+t_0A_3+t_4A_4-t_4A_5)A_4 \\
&= -t_0t_4^{-1}A_4-t_1t_4^{-1}A_1A_4-t_2t_4^{-1}A_2A_4-t_0t_4^{-1}A_3A_4-A_4^2 \\
&\quad +A_4A_5 \\
&= rA_0-rA_3+\left(-\frac{t_0}{t_4}-\frac{k_1}{2}\frac{t_1}{t_4}-\frac{k_2}{2}\frac{t_2}{t_4}\right)A_4 \\
&\quad +\left(-\frac{t_1}{t_2}\frac{k_1}{2}-\frac{t_2}{t_4}\frac{k_2}{2}-\frac{t_0}{t_4}\right)A_5.
\end{aligned}
$$

$$
\begin{aligned}
t_0W_{-}^{T}\circ(W(W_{-}\circ A_4)) &= (A_0+t_0t_1^{-1}A_1+t_0t_2^{-1}A_2+A_3+t_0t_4^{-1}A_4-t_0t_4^{-1}A_5) \\
&\quad \circ\left(rA_0-rA_3+\left(-\frac{t_0}{t_4}-\frac{k_1}{2}\frac{t_1}{t_4}-\frac{k_2}{2}\frac{t_2}{t_4}\right)A_4\right. \\
&\quad \left.+\left(-\frac{t_1}{t_4}\frac{k_1}{2}-\frac{t_2}{t_4}\frac{k_2}{2}-\frac{t_0}{t_4}\right)A_5\right) \\
&= rA_0-rA_3-\frac{t_0}{t_4}\left(\frac{t_0}{t_4}+\frac{k_1}{2}\frac{t_1}{t_4}+\frac{k_2}{2}\frac{t_2}{t_4}\right)A_4 \\
&\quad +\frac{t_0}{t_4}\left(\frac{t_0}{t_4}+\frac{k_1}{2}\frac{t_1}{t_4}+\frac{k_2}{2}\frac{t_2}{t_4}\right)A_5
\end{aligned}
$$

$$=r(A_0-A_3)+\frac{\epsilon(t_0-t_1)(-t_0t_1+\epsilon)}{t_0t_4^2(t_1^2+\epsilon)}(A_4-A_5).$$

$\underline{\Psi(A_5)=t_0W\_^T\circ(W(W\_\circ A_5))}$：

$$W\_\circ A_5=t_4^{-1}A_5,$$

$$\begin{aligned}W(W\_\circ A_5)&=t_4^{-1}(t_0A_0+t_1A_1+t_2A_2+t_0A_3+t_4A_4-t_4A_5)A_5\\&=t_0t_4^{-1}A_5+t_1t_4^{-1}A_1A_5-t_2t_4^{-1}A_2A_5+t_0t_4^{-1}A_3A_5+A_4A_5-A_5^2\\&=rA_0-rA_3+\left(\frac{t_0}{t_4}+\frac{k_1}{2}\frac{t_1}{t_4}+\frac{k_2}{2}\frac{t_2}{t_4}\right)A_4\\&\quad+\left(\frac{t_0}{t_4}+\frac{k_1}{2}\frac{t_1}{t_4}+\frac{k_2}{2}\frac{t_2}{t_4}\right)A_5.\end{aligned}$$

$$\begin{aligned}t_0W\_^T\circ(W(W\_\circ A_5))&=(A_0+t_0t_1^{-1}A_1+t_0t_2^{-1}A_2+A_3+t_0t_4^{-1}A_4-t_0t_4^{-1}A_5)\\&\quad\circ\left(rA_0-rA_3+\left(\frac{t_0}{t_4}+\frac{k_1}{2}\frac{t_1}{t_4}+\frac{k_2}{2}\frac{t_2}{t_4}\right)A_4\right.\\&\quad\left.+\left(\frac{t_0}{t_4}+\frac{k_1}{2}\frac{t_1}{t_4}+\frac{k_2}{2}\frac{t_2}{t_4}\right)A_5\right)\\&=rA_0-rA_3-\frac{t_0}{t_4}\left(\frac{t_0}{t_4}+\frac{k_1}{2}\frac{t_1}{t_4}+\frac{k_2}{2}\frac{t_2}{t_4}\right)A_4\\&\quad+\frac{t_0}{t_4}\left(\frac{t_0}{t_4}+\frac{k_1}{2}\frac{t_1}{t_4}+\frac{k_2}{2}\frac{t_2}{t_4}\right)A_5\\&=r(A_0-A_3)-\frac{\epsilon(t_0-t_1)(-t_0t_1+\epsilon)}{t_0t_4^2(t_1^2+\epsilon)}(A_4-A_5).\end{aligned}$$

We now show that Ψ is a duality.

Checking (7), i.e., $\Psi^2(M)=4rM^T$ for every $M\in\mathcal{A}$, amounts to checking that $P^2=4rR$, where R is the matrix of the transposition operator in the basis $\{A_i|i=0,\ldots,5\}$. This is an easy computation.

To verify (8), we shall check that $\Psi(A_iA_j)=\Psi(A_i)\circ\Psi(A_j)$ for $i,j\in\{0,\ldots,5\}$. To check this, we use the Mathematical Software “Maple”:

```
restart;
```

```
with(LinearAlgebra): interface(rtablesize=infinity):
PP := Matrix( [
[1,
t1*(e*t0^2+1)*(-t0*t1+e)*(t1^3+e*t0)/(e*t0^2*(t1^4-1)*
(t1^2+e)),
-t1*(t0^2+e)*(t0*t1^3+1)*(t1-t0)/(t0^2*(t1^4-1)*(t1^2
+e))],
[1,-t1*(t0^2*t1^2-1)/(t0*(t1^4-1)),(t0*t1^3+1)*(t0-t1)/
(t0*(t1^4-1))],
[1,-(e*t0^2*t1+t0*(t1^4-1)-e*t1^3)/(t0*(t1^4-1)),
e*t1*(t0^2-t1^2)/(t0*(t1^4-1))]] );
t2:=e/t1;E:=(t0-t1)*(t0*t1-e)/(e*t0*(t1^2+e));
e:=-1;
for i1 from 1 to 3 do
for i2 from 1 to 3 do
for i3 from 1 to 3 do
pp[i1-1,i2-1,i3-1]:=
( PP[1,i1]*PP[1,i2]/(E^2) )
*add( (1/( PP[1,y]^2 ))*PP[i1,y]*PP[i2,y]*PP[i3,y], y=
1..3 );
od; od; od;
p11:=factor( t0/t1*(2*t0/t1+2*pp[1,1,1]+t2/t1*2*pp[1,
2,1]) );
p12:=factor( t0/t1*(t1/t2*2*pp[1,2,1]+2*pp[2,2,1]) );
p21:=factor( t0/t2*(2*pp[1,1,2]+t2/t1*2*pp[1,2,2]) );
p22:=factor( t0/t2*(2*t0/t2+t1/t2*2*pp[1,2,2]+2*pp[2,
2,2]) );
t4:=(1/2)*sqrt(2)+(1/2*I)*sqrt(2);
```

```
P:＝Matrix( [
[1,2*PP[1,2],2*PP[1,3],1,n,n],
[1,p11,p12,1,0,0],
[1,p21,p22,1,0,0],
[1,2*PP[1,2],2*PP[1,3],1,-n,-n],
[1,0,0,-1,t0/t4*(t0/t4+PP[1,2]*t1/t4+PP[1,3]*t2/t4),
-t0/t4*(t0/t4+PP[1,2]*t1/t4+PP[1,3]*t2/t4)],
[1,0,0,-1,-t0/t4*(t0/t4+PP[1,2]*t1/t4+PP[1,3]*t2/t4),
t0/t4*(t0/t4+PP[1,2]*t1/t4+PP[1,3]*t2/t4)]
]):
Q:＝Matrix( [
[1,2*PP[1,2],2*PP[1,3],1,n,n],
[1,p11,p12,1,0,0],
[1,p21,p22,1,0,0],
[1,2*PP[1,2],2*PP[1,3],1,-n,-n],
[1,0,0,-1,-t0/t4*(t0/t4+PP[1,2]*t1/t4+PP[1,3]*t2/t4),
t0/t4*(t0/t4+PP[1,2]*t1/t4+PP[1,3]*t2/t4)],
[1,0,0,-1,t0/t4*(t0/t4+PP[1,2]*t1/t4+PP[1,3]*t2/t4),
-t0/t4*(t0/t4+PP[1,2]*t1/t4+PP[1,3]*t2/t4)]
]):
n:＝E^2; UU:＝MatrixMatrixMultiply(P, Q):
for i1 from 1 to 6 do
for i2 from i1 to 6 do
  for l from 0 to 5 do
  print( [i1-1,i2-1,l],
  factor(P[l+1,i1]*P[l+1,i2]-add( pt[i1-1,i2-1,k-1]*
  P[l+1,k],k＝1..6 ) ));
end do;
```

```
end do; end do;
```

□

Assume that $p^{c}_{a,b}\neq 0$. Define

$$p^{ijk}_{abc}(\alpha,\beta,\gamma)=|\{y\in X\,|\,(\alpha,y)\in R_i,\ (\beta,y)\in R_j,\ (\gamma,y)\in R_k\}|.$$

These numbers usually depend on the choice of $\alpha,\beta,\gamma\in X$. If $p^{ijk}_{abc}(\alpha,\beta,\gamma)$ is independent of the choice of $\alpha,\beta,\gamma\in X$, then an association scheme is called a *triply regular.* Then (2) is written by

$$\sum_{i,j,k=0}^{d} p^{ijk}_{abc}(\alpha,\beta,\gamma)\frac{t_i t_j}{t_k}=D\frac{t_a}{t_c t_{b'}}. \tag{27}$$

By Lemma 13, we have $p^{5}_{14}=\tilde{k}_1\neq 0$. We want to determine $p^{ijk}_{abc}(\alpha,\beta,\gamma)$. Them, we have the following:

Lemma 15. *Let* $a_{\alpha\beta\gamma}=p^{114}_{145}(\alpha,\beta,\gamma)$, $e_{\alpha\beta\gamma}=p^{441}_{145}(\alpha,\beta,\gamma)$ *be nonnegative integers. Then, for* $i,j,k\in\{0,\ldots,5\}$, $p^{ijk}_{145}(\alpha,\beta,\gamma)$ *are given by the following:*

i	j	k	$p^{ijk}_{145}(\alpha,\beta,\gamma)$
0	1	4	1
1	0	5	1
1	1	4	$a_{\alpha\beta\gamma}$
1	1	5	$p^{1}_{11}-a_{\alpha\beta\gamma}$
1	2	4	$\frac{k_1}{2}-a_{\alpha\beta\gamma}-1$
1	2	5	$p^{1}_{12}-\frac{k_1}{2}+a_{\alpha\beta\gamma}+1$
1	3	4	1
2	1	4	$\frac{k_1}{2}-a_{\alpha\beta\gamma}-1$
2	1	5	$p^{1}_{22}-\frac{k_1}{2}+a_{\alpha\beta\gamma}+1$
2	2	4	$\frac{k_2}{2}-\frac{k_1}{2}+a_{\alpha\beta\gamma}+1$

2	2	5	$p^{1}_{12}-\frac{k_2}{2}+\frac{k_1}{2}-a_{\alpha\beta\gamma}-1$
3	1	5	1
4	4	1	$e_{\alpha\beta\gamma}$
4	4	2	$\frac{r}{2}-e_{\alpha\beta\gamma}$
4	5	1	$\frac{k_1}{2}-e_{\alpha\beta\gamma}$
4	5	2	$\frac{r}{2}-\frac{k_1}{2}+e_{\alpha\beta\gamma}-1$
4	5	3	1
5	4	0	1
5	4	1	$\frac{k_1}{2}-e_{\alpha\beta\gamma}$
5	4	2	$\frac{r}{2}-\frac{k_1}{2}+e_{\alpha\beta\gamma}-1$
5	5	1	$e_{\alpha\beta\gamma}$
5	5	2	$\frac{r}{2}-e_{\alpha\beta\gamma}$

Proof. In what follows, as a matter of convenience, we set

$$a=p^{114}_{145}(\alpha, \beta, \gamma),$$

$$b=p^{124}_{145}(\alpha, \beta, \gamma),$$

$$c=p^{214}_{145}(\alpha, \beta, \gamma),$$

$$d=p^{224}_{145}(\alpha, \beta, \gamma),$$

$$e=p^{441}_{145}(\alpha, \beta, \gamma),$$

$$f=p^{451}_{145}(\alpha, \beta, \gamma),$$

$$g=p^{541}_{145}(\alpha, \beta, \gamma),$$

$$h=p^{551}_{145}(\alpha, \beta, \gamma).$$

The procedure is as follows:

First step: Let i be given in $\{0, \ldots, 5\}$. We consider the possibilities of

$j \in \{0, \ldots, 5\}$ such that $p^1_{ij'} \neq 0$. Next, we consider the possibilities of $k \in \{0, \ldots, 5\}$ such that $p^5_{ik'} \neq 0$, $p^4_{ik'} \neq 0$.

Let $i=0$. Then $j=1$ and $k=4$.

Let $i=1$. Then $j=0, 1, 2, 3$, and $k=4, 5$. Then, $p^{1jk}_{145}(\alpha, \beta, \gamma)$ are given by using a, b as follows:

i	j	k	$p^{1jk}_{145}(\alpha, \beta, \gamma)$
1	0	5	1
1	1	4	a
1	1	5	$p^1_{11}-a$
1	2	4	b
1	2	5	$p^1_{12}-b$
1	3	4	1

Let $i=2$. Then $j=1, 2$, and $k=4, 5$. Then, $p^{2jk}_{145}(\alpha, \beta, \gamma)$ are given by using c, d as follows:

i	j	k	$p^{2jk}_{145}(\alpha, \beta, \gamma)$
2	1	4	c
2	1	5	$p^1_{12}-c$
2	2	4	d
2	2	5	$p^1_{22}-d$

Let $i=3$. Then $j=1$, $k=5$.

Let $i=4$. Then $j=4, 5$. Then the possibilities of k are $k=1, 2, 3$. Then $p^{4jk}_{145}(\alpha, \beta, \gamma)$ are given by using e, f as follows:

i	j	k	$p^{4jk}_{145}(\alpha, \beta, \gamma)$
4	4	1	e
4	4	2	$\frac{r}{2}-e$
4	5	1	f

4	5	2	$\frac{r}{2}-f-1$
4	5	3	1

Let $i=5$. Then $j=4, 5$. Then the possibilities of k are $k=0, 1, 2$. Then, $p^{5jk}_{145}(\alpha, \beta, \gamma)$ are given by using g, h as follows:

i	j	k	$p^{5jk}_{145}(\alpha, \beta, \gamma)$
5	4	0	1
5	4	1	g
5	4	2	$\frac{r}{2}-1-g$
5	5	1	h
5	5	2	$\frac{r}{2}-h$

From the above, we have the following:

i	j	k	$p^{ijk}_{145}(\alpha, \beta, \gamma)$
1	0	5	1
1	1	4	a
1	1	5	$p^1_{11}-a$
1	2	4	b
1	2	5	$p^1_{12}-b$
1	3	4	1
2	1	4	c
2	1	5	$p^1_{12}-c$
2	2	4	d
2	2	5	$p^1_{22}-d$
3	1	5	1
4	4	1	e
4	4	2	$\frac{r}{2}-e$

4	5	1	f
4	5	2	$\frac{r}{2}-f-1$
4	5	3	1
5	4	0	1
5	4	1	g
5	4	2	$\frac{r}{2}-1-g$
5	5	1	h
5	5	2	$\frac{r}{2}-h$

Second step: Using the above table, for given $i \in \{0, \dots, 5\}$ we change the roles of j, k Then we have the following:

i	j	k	$p_{145}^{ijk}(\alpha, \beta, \gamma)$
0	4	1	1
1	4	1	a
1	4	2	b
1	4	3	1
1	5	0	1
1	5	1	$p_{11}^{1}-a$
1	5	2	$p_{12}^{1}-b$
2	4	1	c
2	4	2	d
2	5	1	$p_{12}^{1}-c$
2	5	2	$p_{22}^{1}-d$
3	5	1	1
4	1	4	e
4	1	5	f
4	2	4	$\frac{r}{2}-e$

4	2	5	$\frac{r}{2}-f-1$
4	3	5	1
5	0	4	1
5	1	4	g
5	1	5	h
5	2	4	$\frac{r}{2}-1-g$
5	2	5	$\frac{r}{2}-h$

In this table, we consider $p_{ik'}^{5}$. Then we have the following:

$$b=\frac{k_1}{2}-a-1,$$

$$d=\frac{k_2}{2}-c,$$

$$f=\frac{k_1}{2}-e,$$

$$h=\frac{k_1}{2}-g.$$

Therefore, we have the following:

i	j	k	$p_{145}^{ijk}(\alpha, \beta, \gamma)$
0	1	4	1
1	0	5	1
1	1	4	a
1	1	5	$p_{11}^{1}-a$
1	2	4	$\frac{k_1}{2}-a-1$
1	2	5	$p_{12}^{1}-\frac{k_1}{2}+a+1$
1	3	4	1

2	1	4	c
2	1	5	$p^1_{12}-c$
2	2	4	$\frac{k_2}{2}-c$
2	2	5	$p^1_{22}-\frac{k_2}{2}+c$
3	1	5	1
4	4	1	e
4	4	2	$\frac{r}{2}-e$
4	5	1	$\frac{k_1}{2}-e$
4	5	2	$\frac{r}{2}-\frac{k_1}{2}+e-1$
4	5	3	1
5	4	0	1
5	4	1	g
5	4	2	$\frac{r}{2}-1-g$
5	5	1	$\frac{k_1}{2}-g$
5	5	2	$\frac{r}{2}-\frac{k_1}{2}+g$

Last step: Using the above table, we change the roles of i, j, k. Then we have the following:

i	j	k	$p^{ijk}_{145}(\alpha, \beta, \gamma)$
0	5	1	1
1	4	0	1
1	4	1	a
1	4	2	c
1	5	1	$p^1_{11}-a$

1	5	2	$p^1_{12}-c$
1	5	3	1
2	4	1	$\frac{k_1}{2}-a-1$
2	4	2	$\frac{k_2}{2}-c$
2	5	1	$p^1_{12}-\frac{k_1}{2}+a+1$
2	5	2	$p^1_{22}-\frac{k_2}{2}+c$
3	4	1	1
4	0	5	1
4	1	4	e
4	1	5	g
4	2	4	$\frac{r}{2}-e$
4	2	5	$\frac{r}{2}-1-g$
5	1	4	$\frac{k_1}{2}-e$
5	1	5	$\frac{k_1}{2}-g$
5	2	4	$\frac{r}{2}-\frac{k_1}{2}+e-1$
5	2	5	$\frac{r}{2}-\frac{k_1}{2}+g$
5	3	4	1

In this table, we consider $p^4_{jk'}$. Then we have the following:

$$c=\frac{k_1}{2}-a-1,$$

$$g=\frac{k_1}{2}-e.$$

Therefore, we have the assertion.

Using Lemma 15, we calculate (2) in the below:

Since $p^5_{14} \neq 0$, we choose three points $\alpha, \beta, \gamma \in X$ such that

$$(\alpha, \beta) \in R_1, \ (\beta, \gamma) \in R_4 (\alpha, \gamma) \in R_5.$$

Then we have the following:

Lemma 16. *The triple-intersection numbers* $a_{\alpha\beta\gamma}$, $e_{\alpha\beta\gamma}$ *defined by Lemma 15 are triply-regular, and*

$$a_{\alpha\beta\gamma} = \frac{p^1_{11}}{2},$$

$$e_{\alpha\beta\gamma} = -\frac{\epsilon t_1 (t_0 + t_1)(t_0 t_1 - \epsilon)(t_0 - t_1)^2}{2 t_0^2 (t_1^2 + \epsilon)^2 (t_1^2 - \epsilon)}.$$

Proof. Let $\alpha \in Y_1$, $\beta \in Y_2$, $\gamma \in Y_3$ in (18). Then

$$\begin{aligned}
(2) \iff & -\sum_{y \in Y_1} \frac{A(\alpha, y) A(\beta, y)}{B^T(\gamma, y)} + \sum_{y \in Y_2} \frac{A(\alpha, y) A(\beta, y)}{B^T(\gamma, y)} \\
& -\sum_{y \in Y_3} \frac{B(\alpha, y) B(\beta, y)}{C(\gamma, y)} - \sum_{y \in Y_4} \frac{B(\alpha, y) B(\beta, y)}{C(\gamma, y)} = 2D \frac{t_1}{t_5^2} \\
\iff & -2 \sum_{y \in Y_3} \frac{B(\alpha, y) B(\beta, y)}{C(\gamma, y)} = 2D \frac{t_1}{t_5^2} \\
\iff & \sum_{y \in Y_3} \frac{B(\alpha, y) B(\beta, y)}{C(\gamma, y)} = -D \frac{t_1}{t_5^2} \\
\iff & (21).
\end{aligned}$$

From this calculation, we have

$$\sum_{y \in Y_1} \frac{A(\alpha, y) A(\beta, y)}{B^T(\gamma, y)} = 0, \tag{28}$$

$$\sum_{y \in Y_3} \frac{B(\alpha, y) B(\beta, y)}{C(\gamma, y)} = -D \frac{t_1}{t_5^2}. \tag{29}$$

On the other hand, we calculate (27) by using Lemma 15:

$$(27) \iff \sum_{i,\,j,\,k=0,\,1,\,2,\,3,\,4,\,5} p_{145}^{ijk}(\alpha,\beta,\gamma)\frac{t_i t_j}{t_k} = 2D\frac{t_1}{t_4^2}$$

$$\iff \sum_{\substack{i,\,j=0,1,2,3\\ k=4,5}} p_{145}^{ijk}(\alpha,\beta,\gamma)\frac{t_i t_j}{t_k} + \sum_{\substack{i,\,j=4,5\\ k=0,1,2,3}} p_{145}^{ijk}(\alpha,\beta,\gamma)\frac{t_i t_j}{t_k} = 2D\frac{t_1}{t_4^2}$$

Combining (28) and (29), we have the next correspondence:

$$\sum_{y\in Y_1}\frac{A(\alpha,y)A(\beta,y)}{B^T(\gamma,y)} = 0 \iff \sum_{\substack{i,\,j=0,1,2,3\\ k=4,5}} p_{145}^{ijk}(\alpha,\beta,\gamma)\frac{t_i t_j}{t_k} = 0, \tag{30}$$

$$\sum_{y\in Y_3}\frac{B(\alpha,y)B(\beta,y)}{C(\gamma,y)} = -D\frac{t_1}{t_5^2} \iff \sum_{\substack{i,\,j=4,5\\ k=0,1,2,3}} p_{145}^{ijk}(\alpha,\beta,\gamma)\frac{t_i t_j}{t_k} = 2D\frac{t_1}{t_4^2}. \tag{31}$$

By Lemma 15, we have

$$\begin{aligned}
(30) \iff & \sum_{\substack{i,\,j=0,1,2,3\\ k=4,5}} p_{145}^{ijk}(\alpha,\beta,\gamma)\frac{t_i t_j}{t_k} \\
&= a_{\alpha\beta\gamma}\frac{t_1^2}{t_4} + (p_{11}^1 - a_{\alpha\beta\gamma})\frac{t_1^2}{t_5} + \left(\frac{k_1}{2} - a_{\alpha\beta\gamma} - 1\right)\frac{t_1 t_2}{t_4} \\
&\quad + \left(p_{12}^1 - \frac{k_1}{2} + a_{\alpha\beta\gamma} + 1\right)\frac{t_1 t_2}{t_5} + \left(\frac{k_1}{2} - a_{\alpha\beta\gamma} - 1\right)\frac{t_1 t_2}{t_4} \\
&\quad + \left(p_{12}^1 - \frac{k_1}{2} + a_{\alpha\beta\gamma} + 1\right)\frac{t_1 t_2}{t_5} + \left(\frac{k_2}{2} - \frac{k_1}{2} + a_{\alpha\beta\gamma} + 1\right)\frac{t_2^2}{t_4} \\
&\quad + \left(p_{22}^1 - \frac{k_2}{2} + \frac{k_1}{2} - a_{\alpha\beta\gamma} - 1\right)\frac{t_2^2}{t_5} \\
&= a\frac{t_1^2}{t_4} - (p_{11}^1 - a_{\alpha\beta\gamma})\frac{t_1^2}{t_4} + \left(\frac{k_1}{2} - a_{\alpha\beta\gamma} - 1\right)\frac{t_1 t_2}{t_4} \\
&\quad - \left(p_{12}^1 - \frac{k_1}{2} + a_{\alpha\beta\gamma} + 1\right)\frac{t_1 t_2}{t_4} + \left(\frac{k_1}{2} - a_{\alpha\beta\gamma} - 1\right)\frac{t_1 t_2}{t_4} \\
&\quad - \left(p_{12}^1 - \frac{k_1}{2} + a_{\alpha\beta\gamma} + 1\right)\frac{t_1 t_2}{t_4} + \left(\frac{k_2}{2} - \frac{k_1}{2} + a_{\alpha\beta\gamma} + 1\right)\frac{t_2^2}{t_4}
\end{aligned}$$

$$-\left(p_{22}^{1}-\frac{k_2}{2}+\frac{k_1}{2}-a_{\alpha\beta\gamma}-1\right)\frac{t_2^2}{t_4}$$

$$=(2a_{\alpha\beta\gamma}-p_{11}^{1})\frac{t_1^2}{t_4}+(2k_1-2p_{12}^{1}-4a_{\alpha\beta\gamma}-4)\frac{t_1t_2}{t_4}$$

$$+(k_2-k_1-p_{22}^{1}+2a_{\alpha\beta\gamma}+2)\frac{t_2^2}{t_4}$$

$$=(2a_{\alpha\beta\gamma}+p_{11}^{1})\left(\frac{t_1^2}{t_4}-2\frac{t_1t_2}{t_4}+\frac{t_2^2}{t_4}\right)$$

$$=(2a_{\alpha\beta\gamma}-p_{11}^{1})\frac{(t_1-t_2)^2}{t_4}$$

$$=0.$$

Since $t_1\neq t_2$, we have

$$a_{\alpha\beta\gamma}=\frac{p_{11}^{1}}{2}.$$

LHS of (31)

$$\Longleftrightarrow\ 2\left(e_{\alpha\beta\gamma}\frac{t_4^2}{t_1}+\left(\frac{m}{2}-e_{\alpha\beta\gamma}\right)\frac{t_4^2}{t_2}-\left(\frac{k_1}{2}-e_{\alpha\beta\gamma}\right)\frac{t_4^2}{t_1}\right.$$

$$\left.-\left(\frac{m}{2}-\frac{k_1}{2}+e_{\alpha\beta\gamma}-1\right)\frac{t_4^2}{t_2}-\frac{t_4^2}{t_0}\right)$$

$$=2\left(\left(2e_{\alpha\beta\gamma}-\frac{k_1}{2}\right)\frac{t_4^2}{t_1}+\left(\frac{m}{2}-e_{\alpha\beta\gamma}-\frac{m}{2}+\frac{k_1}{2}-e_{\alpha\beta\gamma}+1\right)\frac{t_4^2}{t_2}-\frac{t_4^2}{t_0}\right)$$

$$=2t_4^2\left(\left(2e_{\alpha\beta\gamma}-\frac{k_1}{2}\right)\frac{1}{t_1}+\left(\frac{k_1}{2}-2e_{\alpha\beta\gamma}+1\right)\frac{1}{t_2}-\frac{1}{t_0}\right)$$

$$=2t_4^2\left(\left(2e_{\alpha\beta\gamma}-\frac{k_1}{2}\right)\left(\frac{1}{t_1}-\frac{1}{t_2}\right)+\frac{1}{t_2}-\frac{1}{t_0}\right)$$

$$=2D\frac{t_1}{t_5^2}$$

$$=2D\frac{t_1}{t_4^2}$$

$\Longleftrightarrow$ RHS of (31)

Therefore, we have

$$t_4^4\left(\left(2e_{\alpha\beta\gamma}-\frac{k_1}{2}\right)\left(\frac{1}{t_1}-\frac{1}{t_2}\right)+\frac{1}{t_2}-\frac{1}{t_0}\right)=Dt_1.$$

From this equation, we have

$$e_{\alpha\beta\gamma}=-\frac{\epsilon t_1(t_0+t_1)(t_0t_1-\epsilon)(t_0-t_1)^2}{2t_0^2(t_1^2+\epsilon)^2(t_1^2-\epsilon)}.$$

Since (2) holds for any α, β, $\gamma\in X$, (2) is written by

$$\sum_{x\in X}\frac{W(\beta,x)W(\gamma,x)}{W(\alpha,x)}=D\frac{W(\beta,\gamma)}{W(\beta,\alpha)W(\alpha,\gamma).} \tag{32}$$

$$\begin{aligned}(32)\iff & -\sum_{y\in Y_1}\frac{A(\beta,y)B^T(\gamma,y)}{A(\alpha,y)}+\sum_{y\in Y_2}\frac{A(\beta,y)B^T(\gamma,y)}{A(\alpha,y)}\\ &\quad-\sum_{y\in Y_3}\frac{B(\beta,y)C(\gamma,y)}{B(\alpha,y)}-\sum_{y\in Y_4}\frac{B(\beta,y)C(\gamma,y)}{B(\alpha,y)}=2D\frac{t_4}{t_1t_5}\\ \iff & -2\sum_{y\in Y}\frac{B(\beta,y)C(\gamma,y)}{B(\alpha,y)}=-2\frac{D}{t_1}\\ \iff & \sum_{y\in Y}\frac{B(\beta,y)C(\gamma,y)}{B(\alpha,y)}=\frac{D}{t_1}.\end{aligned}$$

Therefore, we have

$$\sum_{y\in Y_1}\frac{A(\beta,y)B^T(\gamma,y)}{A(\alpha,y)}=0, \tag{33}$$

$$\sum_{y\in Y}\frac{B(\beta,y)C(\gamma,y)}{B(\alpha,y)}=\frac{D}{t_1}. \tag{34}$$

$$\begin{aligned}(33)\iff & \left(a_{\alpha\beta\gamma}-p_{11}^1+a_{\alpha\beta\gamma}-p_{22}^1+\frac{k_2}{2}-\frac{k_1}{2}+a_{\alpha\beta\gamma}+1\right)t_4\\ &+\left(\frac{k_1}{2}-a_{\alpha\beta\gamma}-1-p_{12}^1+\frac{k_1}{2}-a_{\alpha\beta\gamma}+1\right)\frac{t_2t_4}{t_1}\\ &+\left(\frac{k_1}{2}-a_{\alpha\beta\gamma}-1-p_{12}^1+\frac{k_1}{2}-a_{\alpha\beta\gamma}+1\right)\frac{t_1t_4}{t_2}\end{aligned}$$

$$=(k_2-k_1-p^1_{11}-p^1_{22}+4a_{\alpha\beta\gamma}+2)t_4$$

$$+(k_1-p^1_{12}-2-2a_{\alpha\beta\gamma})\frac{t_2t_4}{t_1}+(k_1-p^1_{12}-2a_{\alpha\beta\gamma}-2)\frac{t_1t_4}{t_2}$$

$$=(p^1_{11}-2a_{\alpha\beta\gamma})\frac{t_4(t_1-t_2)^2}{t_1t_2}$$

$$=0.$$

Since $t_1\neq t_2$, we have

$$a_{\alpha\beta\gamma}=\frac{p^1_{11}}{2}.$$

$$(34)\iff 2\left(-t_0+\left(2e_{\alpha\beta\gamma}-\frac{k_1}{2}\right)t_1+\left(\frac{n}{2}-e_{\alpha\beta\gamma}-\frac{n}{2}+\frac{k_1}{2}-e_{\alpha\beta\gamma}+1\right)t_2\right)$$

$$=2D\frac{t_4}{t_1t_5}=-\frac{2D}{t_1}$$

$$\iff -2\left(t_0+\left(\frac{k_1}{2}-2e_{\alpha\beta\gamma}\right)t_1+\left(2e_{\alpha\beta\gamma}-1-\frac{k_1}{2}\right)t_2\right)=-\frac{2D}{t_1}$$

$$\iff \left(t_0+\left(\frac{k_1}{2}-2e_{\alpha\beta\gamma}\right)t_1+\left(2e_{\alpha\beta\gamma}-1-\frac{k_1}{2}\right)t_2\right)=-\frac{D}{t_1}.$$

From this equation, we have

$$e_{\alpha\beta\gamma}=-\frac{\epsilon t_1(t_0+t_1)(t_0t_1-\epsilon)(t_0-t_1)^2}{2t_0^2(t_1^2+\epsilon)^2(t_1^2-\epsilon)}.$$

□

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